Chapter 6, Efficient Diversification

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Chapter 6 Overview

Diversification

Two Risky Assets

Stocks, Bonds, and Bills

Markowitz Optimization

Diversification and Porftolio Risk

- Market risk
  - risk from marketwide risk sources/general economic conditions, such as business cycles, inflation, interest rates, exchange rates
  - remains even after extensive diversification
  - Also called systematic or nondiversifiable
- Firm-specific risk
  - due to firm-specific influences, such as success in R&D, its management style and philosophy, etc.
  - Risk that can be eliminated by diversification
  - Also called diversifiable or nonsystematic

Figure 6.1 Porfolio Risk and the Number of Stocks

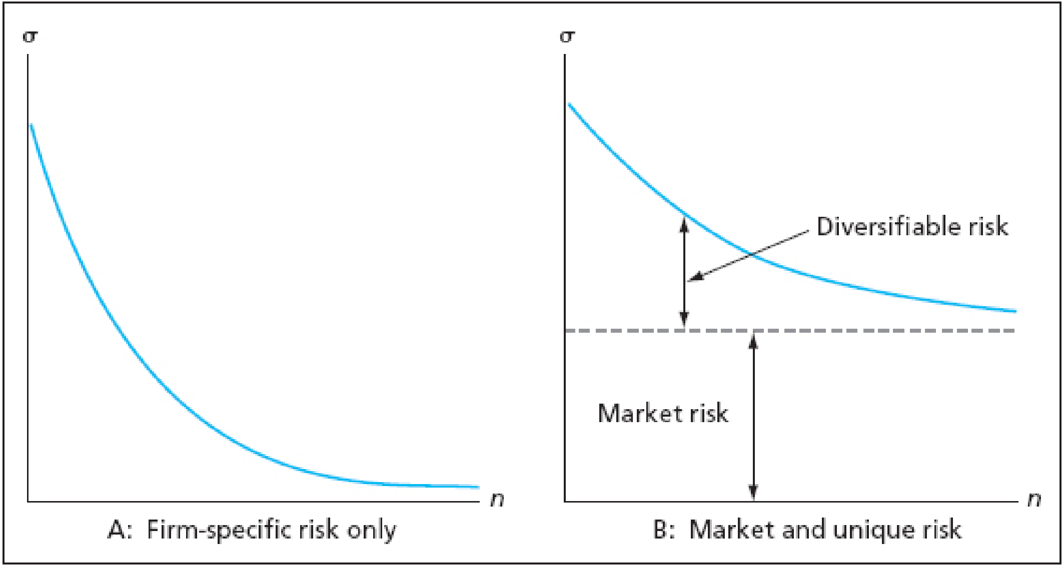
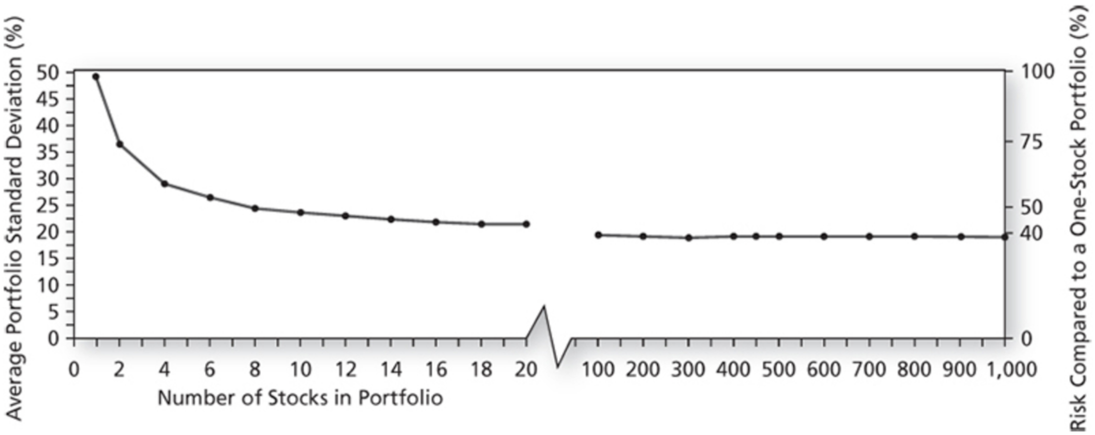


Table Risk Reduction of Equally Weighted Portfolios

Universe Size n	Portfolio Weights $w = 1/n$ (%)	$\rho = 0$ Standard Deviation (%)	$\rho = 0$ Reduction in σ	$\rho = .40$ Standard Deviation (%)	$\rho = .40$ Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	

Figure 6.2 Portfolio Diversification



Portfolio of Two Risky Assets

- Portfolio risk (variance) depends on the correlation between the returns of the assets in the portfolio
- Covariance and the correlation coefficient provide a measure of the way returns of two assets move together (covary)

Capital market expectations for the stock and bond funds

	A	B	C	D	E	F
1	**Spreadsheet 6.1**					
2	**Capital market expectations for stock and bond funds**					
3			Stock Fund		Bond Fund	
4	Scenario	Probability	Rate of Return	Col B x Col C	Rate of Return	Col B x Col E
5	Severe recession	0.05	-37	-1.9	-9	-0.5
6	Mild recession	0.25	-11	-2.8	15	3.8
7	Normal growth	0.40	14	5.6	8	3.2
8	Boom	0.30	30	9.0	-5	-1.5
9	Expected or Mean Return:		SUM:	10.0	SUM:	5.0

http://somfin.gmu.edu/courses/fnan311/excel/Spreadsheet_6_1.xlsx

Variance of Returns

	A	B	C	D	E	F	G	H	I	J
1	**Spreadsheet 6.2**									
2	**Variance of returns**									
3				**Stock Fund**				**Bond Fund**		
4				Deviation				Deviation		
5			Rate	from		Column B	Rate	from		Column B
6			of	Expected	Squared	x	of	Expected	Squared	x
7	Scenario	Prob.	Return	Return	Deviation	Column E	Return	Return	Deviation	Column I
8	Severe recession	0.05	-37	-47	2209	110.45	-9	-14	196	9.80
9	Mild recession	0.25	-11	-21	441	110.25	15	10	100	25.00
10	Normal growth	0.40	14	4	16	6.40	8	3	9	3.60
11	Boom	0.30	30	20	400	120.00	-5	-10	100	30.00
12				Variance = SUM		347.10			Variance:	68.40
13		Standard deviation = SQRT(Variance)				18.63			Std. Dev.:	8.27

http://somfin.gmu.edu/courses/fnan311/excel/Spreadsheet_6_2.xlsx

Variance of Portfolio Returns

	A	B	C	D	E	F	G
1	**Spreadsheet 6.3**						
2	**Performance of a portfolio invested in the stock and bond funds**						
3			Portfolio invested 40% in stock fund and 60% in bond fund				
4			Rate	Column B	Deviation from		Column B
5			of	x	Expected	Squared	x
6	Scenario	Probability	Return	Column C	Return	Deviation	Column F
7	Severe recession	0.05	-20.2	-1.01	-27.2	739.84	36.99
8	Mild recession	0.25	4.6	1.15	-2.4	5.76	1.44
9	Normal growth	0.40	10.4	4.16	3.4	11.56	4.62
10	Boom	0.30	9.0	2.70	2.0	4.00	1.20
11			Expected return:	7.00		Variance:	44.26
12						Standard deviation:	6.65

http://somfin.gmu.edu/courses/fnan311/excel/Spreadsheet_6_3.xlsx

Portfolio of Two Risky Assets: Covariance

- Covariance of returns on bond and equity:

$$Cov(r_D, r_E) = \sum_s p(s)\left[r_D(s) - E(r_D)\right]\left[r_E(s) - E(r_E)\right] \tag{6}$$

Alternatively, knowing the correlation coefficient, ρ:

$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E \tag{7}$$

where

$$\rho_{DE} = \text{Correlation coefficient of returns}$$
$$\sigma_D = \text{Standard deviation of bond returns}$$
$$\sigma_E = \text{Standard deviation of equity returns}$$

Covariance between the returns of the stock and bond funds

	A	B	C	D	E	F
1	**Spreadsheet 6.4**					
2	Covariance between the returns of the stock and bond funds					
3			Deviation from Mean Return		Covariance	
4	Scenario	Probability	Stock Fund	Bond Fund	Product of Dev	Col B x Col E
5	Severe recession	0.05	-47	-14	658	32.9
6	Mild recession	0.25	-21	10	-210	-52.5
7	Normal growth	0.40	4	3	12	4.8
8	Boom	0.30	20	-10	-200	-60.0
9				Covariance =	SUM:	-74.8
10	Correlation coefficient = Covariance/(StdDev(stocks)*StdDev(bonds)) =					-0.49

http://somfin.gmu.edu/courses/fnan311/excel/Spreadsheet_6_4.xlsx

Concept Check

CONCEPT check 6.1

Suppose the rates of return of the bond portfolio in the four scenarios of Spreadsheet 6.1 are –10% in a severe recession, 10% in a mild recession, 7% in a normal period, and 2% in a boom. The stock returns in the four scenarios are –37%, –11%, 14%, and 30%. What are the covariance and correlation coefficient between the rates of return on the two portfolios?

http://somfin.gmu.edu/courses/fnan311/excel/Spreadsheet_6_1.xlsx

Portfolio of Two Risky Assets: Correlation

- Range of values for ρ_{DE}:

$$-1.0 \leq \rho_{DE} \leq +1.0$$

- If $\rho = 1.0$, the securities are pefectly positively correlated
- If $\rho = -1.0$, the securities are pefectly negatively correlated

Table with data for Example 6.1

	A	B	C	D	E	F
1		**Rates of Return**		**Deviations from Average Returns**		Products of
2	**Year**	Stock Fund	Bond Fund	Stock Fund	Bond Fund	Deviations
3	2006	30.17	5.08	20.17	0.08	1.53
4	2007	32.97	7.52	22.97	2.52	57.78
5	2008	21.04	−8.82	11.04	−13.82	−152.56
6	2009	−8.10	5.27	−18.10	0.27	−4.82
7	2010	−12.89	12.20	−22.89	7.20	−164.75
8	2011	−28.53	−7.79	−38.53	−12.79	493.00
9	2012	22.49	6.38	12.49	1.38	17.18
10	2013	12.58	12.40	2.58	7.40	19.05
11	2014	14.81	17.29	4.81	12.29	59.05
12	2015	15.50	0.51	5.50	−4.49	−24.70
13						
14	Average	10.00	5.00	Covariance = average product of deviations:		30.08
15	SD	19.00	8.00	Correlation = Covariance/(SD stocks*SD bonds):		0.20
16						
17	**Excel formulas**					
18	Average	=average(B3:B12)				
19	Std deviation	=stdevp(B3:B12)				
20	Covariance	=covar(B3:B12,C3:C12)				
21	Correlation	=correl(B3:B12,C3:C12)				
22						
23						

http://somfin.gmu.edu/courses/fnan311/excel/ex6p1.csv

Portfolio of Two Risky Assets: Return

- Portfolio return:

$$r_P = w_D r_D + w_E r_E \tag{8}$$

where

$$\begin{aligned} w_D &= \text{Bond weight} \\ r_D &= \text{Bond return} \\ w_E &= \text{Equity weight} \\ r_E &= \text{Equity return} \end{aligned}$$

- Portfolio expected return:

$$E(r_P) = w_D E(r_D) + w_E E(r_E) \tag{9}$$

Portfolio of Two Risky Assets: Risk

- Portfolio variance:

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E) \tag{10}$$

where

$$\begin{aligned} \sigma_D^2 &= \text{Bond variance} \\ \sigma_E^2 &= \text{Equity variance} \\ Cov(r_D, r_E) &= \text{Covariance of returns for bond and equity} \end{aligned}$$

Portfolio of Two Risky Assets: Correlation

- When $\rho_{DE} = 1$, there is no diversification

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \sigma_D \sigma_E \quad (11)$$
$$= (w_D\sigma_D + w_E\sigma_E)^2 \quad (12)$$
$$\sigma_P = w_D\sigma_D + w_E\sigma_E \quad (13)$$

  - σ_P is a weighted average of component standard deviations

- When $\rho_{DE} = -1$, there is potential for perfect hedge

$$\sigma_P^2 = (w_D\sigma_D - w_E\sigma_E)^2 \quad (14)$$

solve for $(w_D\sigma_D - w_E\sigma_E) = 0$:

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D \quad (16)$$

Figure Mean-Variance Criterion

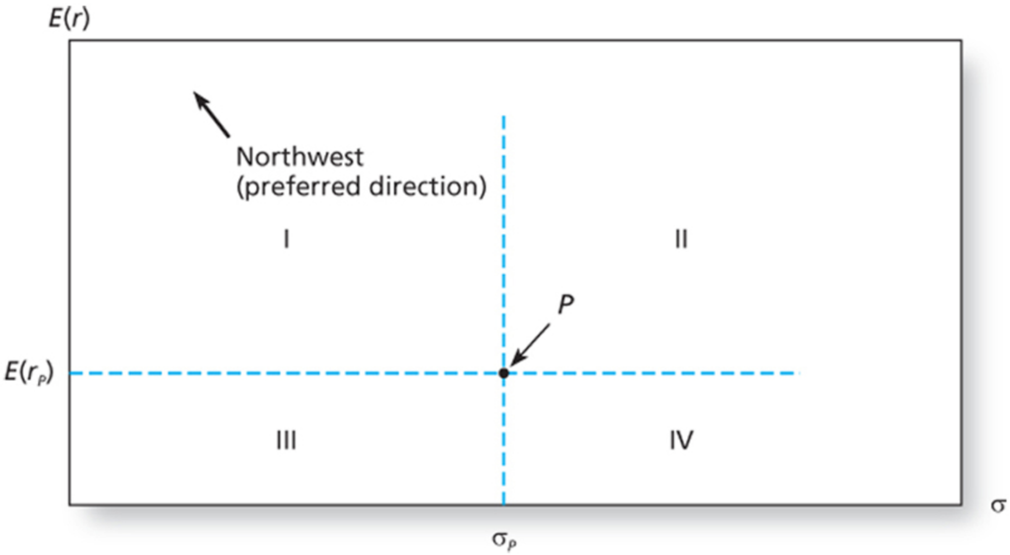
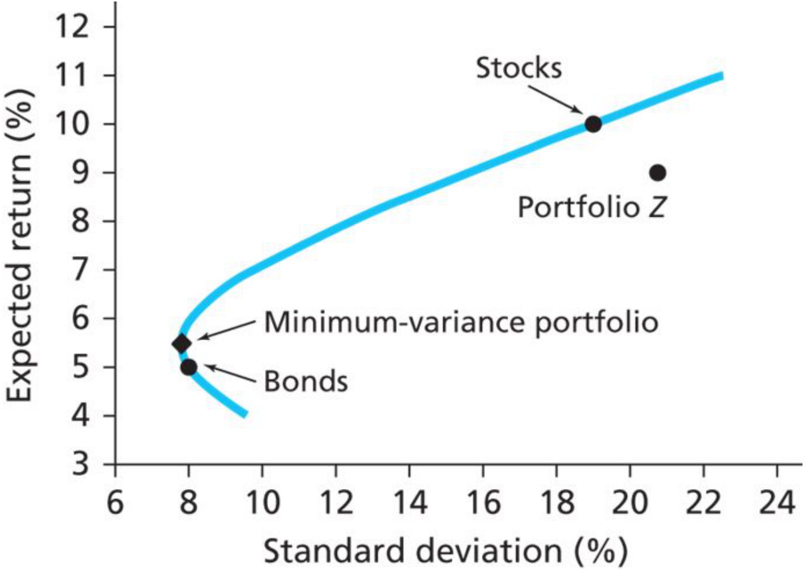


Figure 6.3 The investment opportunity set



	A	B	C	D	E
1	**Spreadsheet 6.5**				
2	**The investment opportunity set with the stock and bond funds**				
3			**Input Data**		
4	$E(r_S)$	$E(r_B)$	σ_S	σ_B	ρ_{BS}
5	10	5	19	8	0.2
6	Portfolio Weights		Expected Return, $E(r_P)$		Std Dev
7	$w_S = 1 - w_B$	w_B	Col A*A3 + Col B*B3		Equation 6.6
8	-0.2	1.2	4.0		9.59
9	-0.1	1.1	4.5		8.62
10	0.0	1.0	5.0		8.00
11	0.0932	0.9068	5.5		7.804
12	0.1	0.9	5.5		7.81
13	0.2	0.8	6.0		8.07
14	0.3	0.7	6.5		8.75
15	0.4	0.6	7.0		9.77
16	0.5	0.5	7.5		11.02
17	0.6	0.4	8.0		12.44
18	0.7	0.3	8.5		13.98
19	0.8	0.2	9.0		15.60
20	0.9	0.1	9.5		17.28
21	1.0	0.0	10.0		19.00
22	1.1	-0.1	10.5		20.75
23	1.2	-0.2	11.0		22.53

http://somfin.gmu.edu/courses/fnan311/excel/Spreadsheet_6_5.xlsx

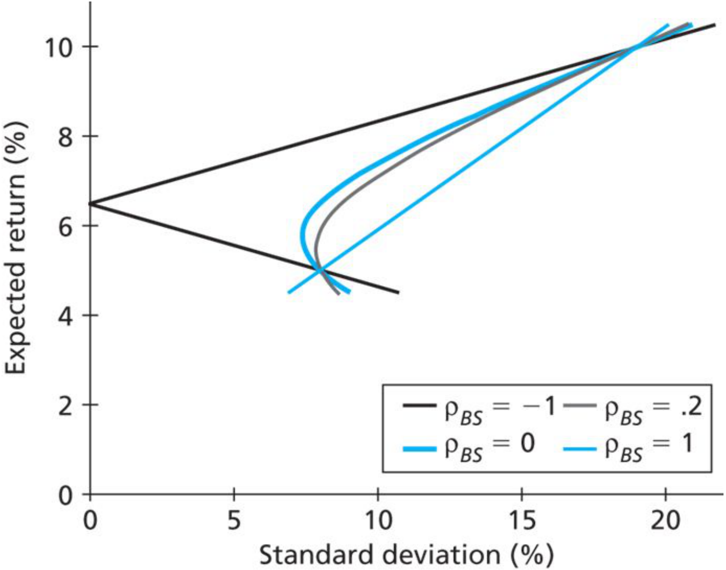
The Minimum Variance Portfolio

- The minimum variance portfolio is the portfolio composed of the risky assets that has the smallest standard deviation; the portfolio with least risk
- For the two-asset case the weights of the min-var portfolio are:

$$w_D^{min} = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)} \qquad w_E^{min} = 1 - w_D^{min}$$

- The amount of possible risk reduction through diversification depends on the correlation:
  - If $\rho = +1.0$, no risk reduction is possible
  - If $\rho = 0$, σ_P may be less than the standard deviation of either component asset
  - If $\rho = -1.0$, a riskless hedge is possible

Figure 6.4 The investment opportunity set with different correlations



Concept check

CONCEPT *check* 6.3

Suppose that for some reason you are *required* to invest 50% of your portfolio in bonds and 50% in stocks. Use the data on mean returns and standard deviations in Spreadsheet 6.5 to answer the following questions.

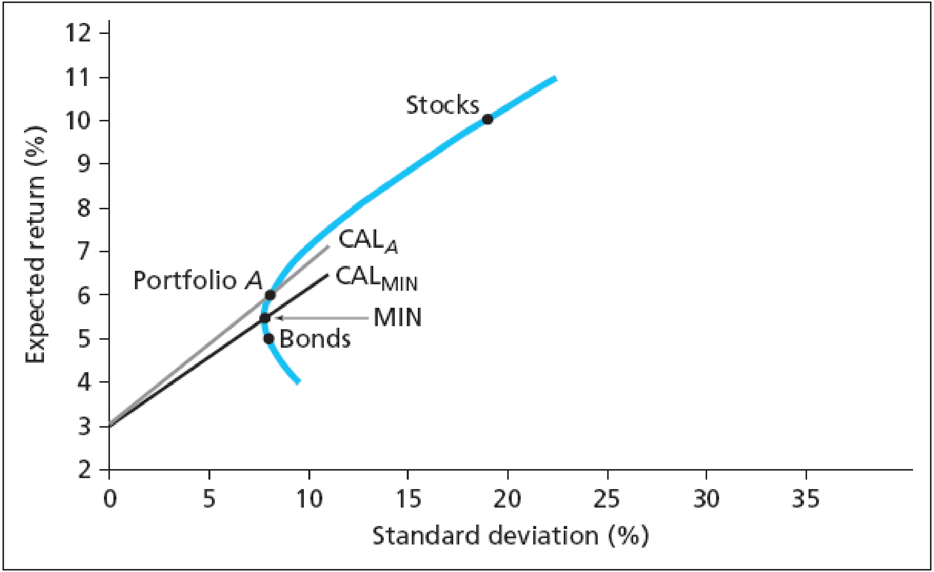
a. If the standard deviation of your portfolio is 10%, what must be the correlation coefficient between stock and bond returns?

b. What is the expected rate of return on your portfolio?

c. Now suppose that the correlation between stock and bond returns is .22 instead of the value you found in part (*a*) but that you are free to choose whatever portfolio proportions you desire. Are you likely to be better or worse off than you were in part (*a*)?

answer

Figure 6.5 The Opportunity Set with Stocks and Bonds



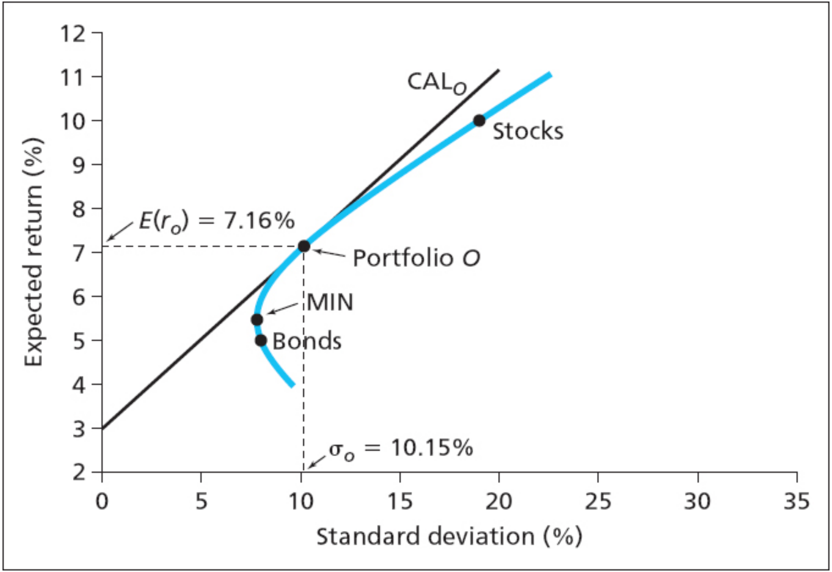
The Sharpe Ratio

- Maximize the slope of the CAL for any possible portfolio, P
- The objective function is the slope:

$$S = \frac{E(r_P) - r_f}{\sigma_P}$$

- The slope is also the Sharpe ratio

Figure 6.6 Optimal Risky Portfolio



Optimal Risky Portfolio Problem

$$\max_{w_i} S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$w_D = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]Cov(R_D, R_E)} \quad (17)$$

$$w_E = 1 - w_D$$

Example Optimal Risky Portfolio

Using our data, the optimal risky portfolio is:

$$
\begin{aligned}
w_D =& \frac{(5-3)\times(19^2)-(10-3)\times(0.2\times 8\times 19)}{(5-3)\times(19^2)+(10-3)\times(8^2)-(5-3+10-3)\times(0.2\times 8\times 19)} = 0.568 \\
w_E =& 1-0.568 = 0.432
\end{aligned}
$$

giving an expected return and standard deviation:

$$
\begin{aligned}
E(r_P) =& (0.568\times 5)+(0.432\times 10) = 7.16\% \\
\sigma_P =& \sqrt{(0.568^2\times 8^2)+(0.432^2\times 19^2)+(2\times 0.568\times 0.432\times 0.2\times 8\times 19)} = 10.15\%
\end{aligned}
$$

and a slope of the CAL(P):

$$
S_P = \frac{7.16-3}{10.15} = 0.41
$$

Example Optimal Complete Portfolio

Now that asset allocation is decided, we can find each investor's optimal capital allocation. An investor with a coefficient of risk aversion A = 5 would take a position in portfolio P (note the use of decimals):

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{0.0716 - 0.03}{5 \times 0.1015^2} = 0.8176 \tag{18}$$

Thus the investor will invest 81.76% of his or her wealth in portfolio P and 19.24% in T-bills. Portfolio P consists of 56.8% in bonds, so the fraction of wealth in bonds will be $y_{w_D} = 0.568 \times 0.8076 = 0.4587$, or 45.87%. Similarly, the investment in stocks will be $y_{w_E} = 0.432 \times 0.8076 = 0.3489$, or 34.89%. The graphical solution of this asset allocation problem is in the Figures on the next two slides.

Figure 6.7 Determining Overall Optimal Portfolio

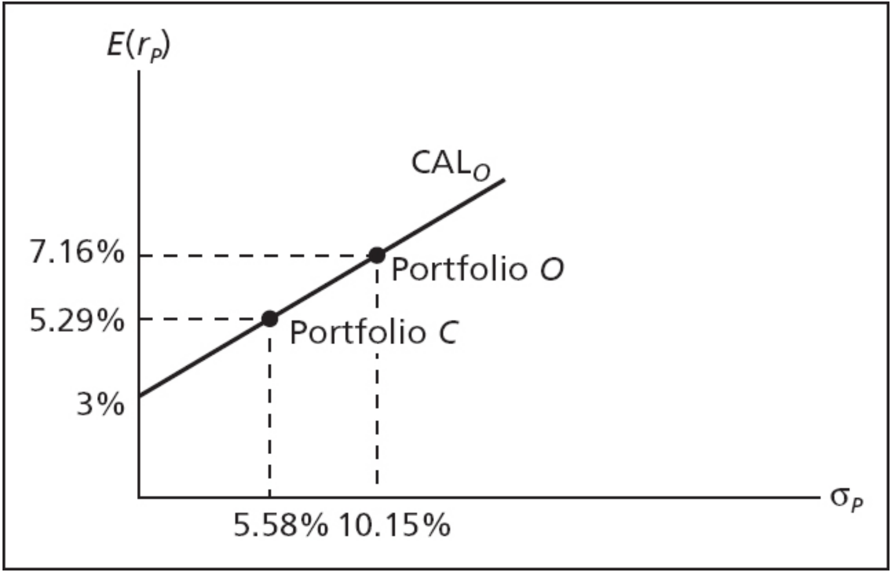
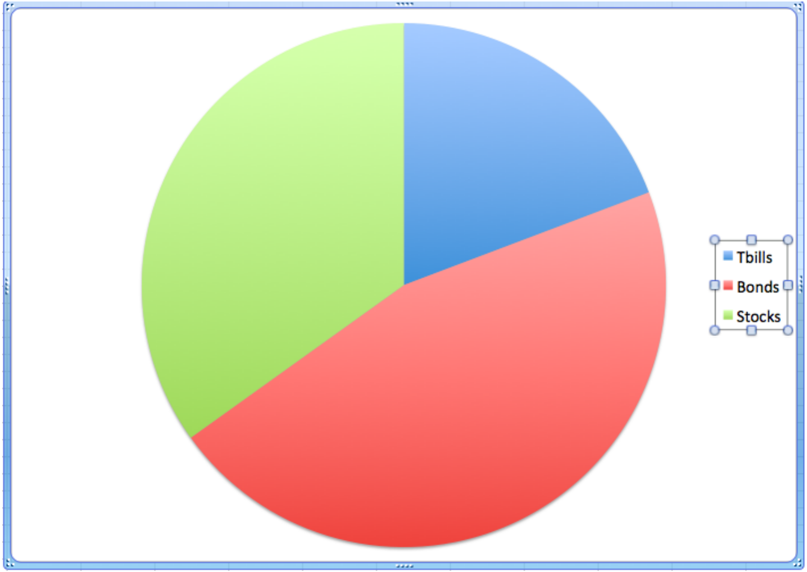


Figure 6.8 The Proportions of the Optimal Complete Portfolio



Concept check

CONCEPT check 6.4

A universe of securities includes a risky stock (*X*), a stock-index fund (*M*), and T-bills. The data for the universe are:

	Expected Return	Standard Deviation
X	15%	50%
M	10	20
T-bills	5	0

The correlation coefficient between *X* and *M* is –.2.

a. Draw the opportunity set of securities *X* and *M.*

b. Find the optimal risky portfolio (*O*), its expected return, standard deviation, and Sharpe ratio. Compare with the Sharpe ratio of *X* and *M.*

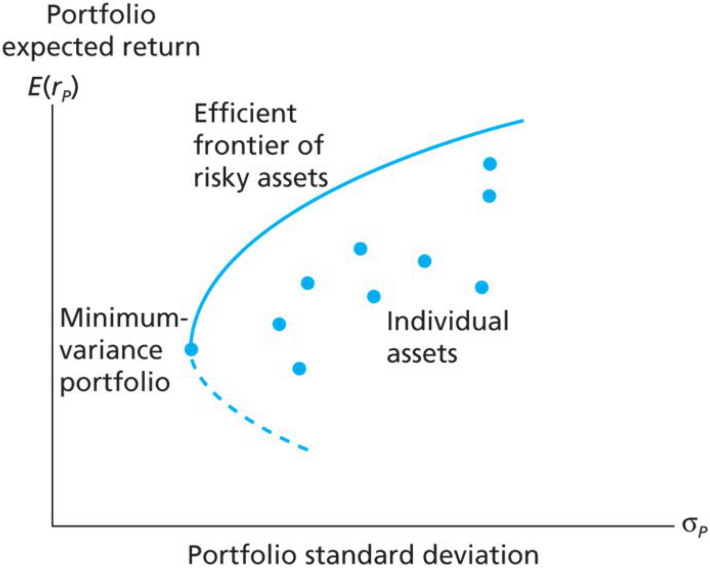
c. Find the slope of the CAL generated by T-bills and portfolio *O.*

d. Suppose an investor places 2/9 (i.e., 22.22%) of the complete portfolio in the risky portfolio *O* and the remainder in T-bills. Calculate the composition of the complete portfolio, its expected return, SD, and Sharpe ratio.

Markowitz Portfolio Optimization Model

- Security selection
  - The first step is to determine the risk-return opportunities available
  - All portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk-return combinations

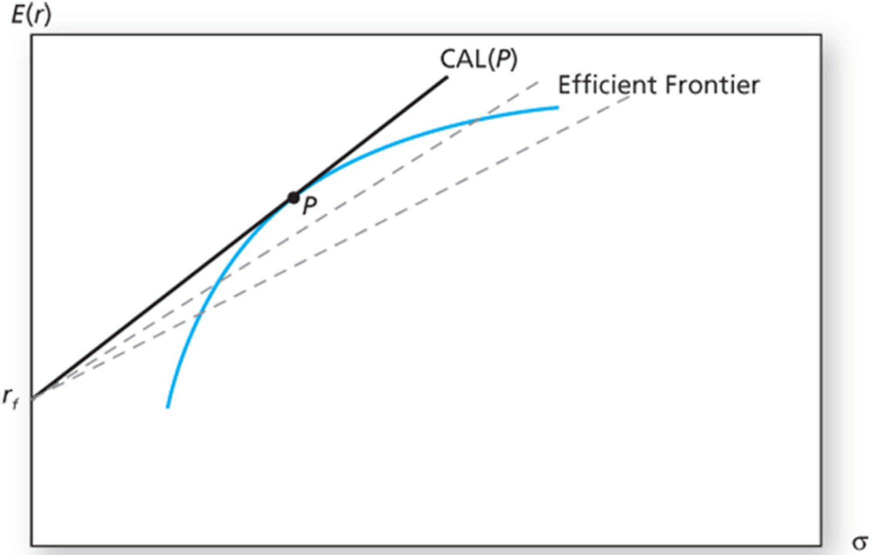
Figure 6.10 The efficient frontier with risk assets



Markowitz Portfolio Optimization Model

- Search for the CAL with the highest reward-to-variability ratio
- Everyone invests in P, regardless of their degree of risk aversion
  - More risk averse investors put more in the risk-free asset
  - Less risk averse investors put more in P

Figure Efficient Frontier of Risky Assets with the Optimal CAL



Markowitz Portfolio Optimization Model

- Capital Allocation and the Separation Property
  - Portfolio choice problem may be separated into two independent tasks
    - Determination of the optimal risky portfolio is purely technical
    - Allocation of the complete portfolio to risk-free versus the risky portfolio depends on personal preference

Figure The Efficient Portfolio Set

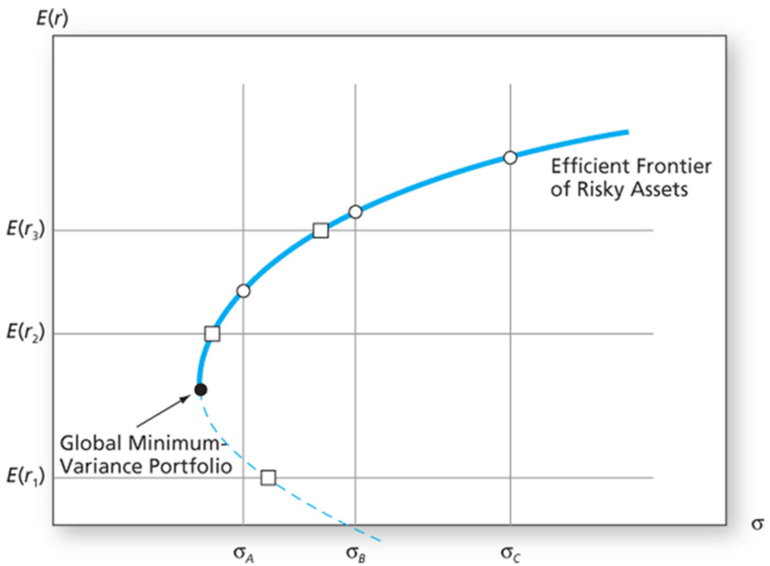
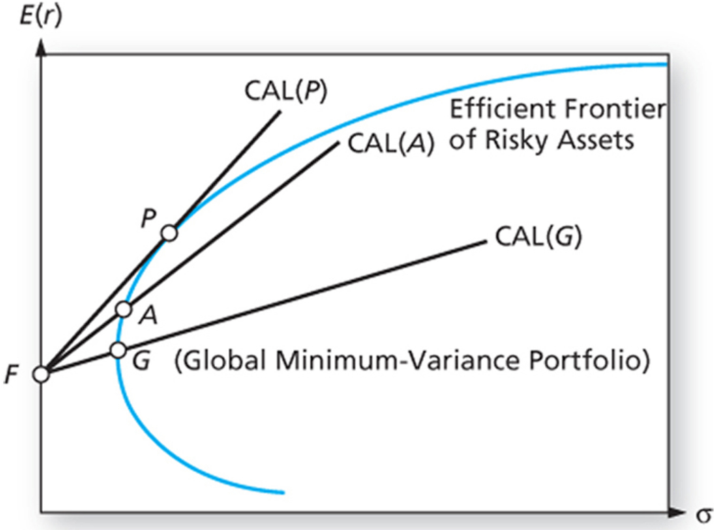


Figure CAL with Various Portfolios from the Efficient Set



The Single-Index Model

- Regression equation, for each date t:

$$R_i(t) = \alpha_i + \beta_i R_M(t) + e_i(t) \tag{8}$$

where

$R_i = r_i - r_f$ is the excess return on a security
$R_M = r_M - r_f$ is the excess return on the market
$\beta_i =$ is the security?s sensitivity to the index

- Expected return-beta relationship

$$E(R_i) = \alpha_i + \beta_i E(R_M) \tag{9}$$

where

$\beta_i E(R_M) =$ systematic risk premium
$\alpha_i =$ nonmarket premium

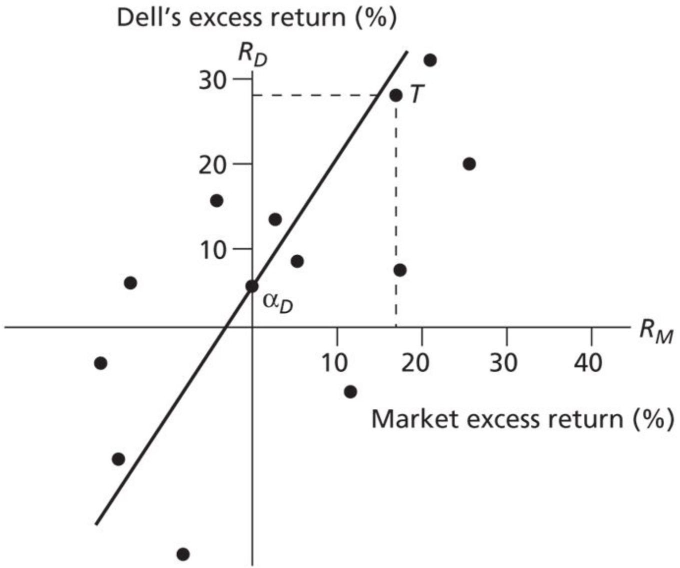
The Single-Index Model

$$
\begin{aligned}
\text{Total Risk} &= \text{Systematic Risk} + \text{Firm-specific risk} \\
\sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_{e_i}^2 \\
\text{Covariance} &= \text{Product of betas} \times \text{Market index risk} \\
Cov(r_i, r_j) &= \beta_i \beta_j \sigma_M^2 \qquad (10)
\end{aligned}
$$

Components of the Index Model: Summary

	Symbol
1. The stock's expected return if the market is neutral, that is, if the market's excess return, $r_M - r_f$, is zero	α_i
2. The component of return due to movements in the overall market; β_i is the security's responsiveness to market movements	$\beta_i(r_M - r_f)$
3. The unexpected component of return due to unexpected events that are relevant only to this security (firm specific)	e_i
4. The variance attributable to the uncertainty of the common macroeconomic factor	$\beta_i^2\sigma_M^2$
5. The variance attributable to firm-specific uncertainty	$\sigma^2(e_i)$

Figure 6.12 Scatter diagram for Dell



Concept check

CONCEPT check 6.6

Interpret the eight scatter diagrams of **Figure 6.13** in terms of systematic risk, diversifiable risk, and the intercept.

FIGURE 6.13 Various scatter diagrams