Chapter 5, Risk and Return

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Chapter Overview

Rates of Return for Different Horizons

The simplest investment is a one time purchase of a financial asset (e.g. a zero-coupon bond) at time zero (now), and receiving a single payoff at a future time T, measured in years. The time interval between 0 and T is the investment horizon.

- Example: Rate of Return on a Zero Coupont Bond, Horizon of T years

$$\text{Par} = \$100$$
$$\text{Maturity} = T$$
$$\text{Price} = P$$

Total risk free return over the holding period:

$$r_f(T) = \frac{100 - P(T)}{P(T)} = \frac{100}{P(T)} - 1 \qquad (5.6)^{1}$$

[1] Equation numbers on the slides correspond to equation numbers in the textbook.

Example 5.2 Rates of return over different horizons

Suppose prices of zero-coupon Treasuries with $100 face value and various maturities are as follows. We find the total return of each security by using Equation 5.6:

Horizon, T	Price, $P(T)$	$[100/P(T)] - 1$	Risk-Free Return for Given Horizon
Half-year	$97.36	100/97.36 − 1 = .0271	$r_f(.5) = 2.71\%$
1 year	$95.52	100/95.52 − 1 = .0469	$r_f(1) = 4.69\%$
25 years	$23.30	100/23.30 − 1 = 3.2918	$r_f(25) = 329.18\%$

Effective Annual Rates

- Holding period rates are usually over different horizons, hence different scale, which makes them unsuitable for comparing investments.
- To compare investments, we need to convert rates to the same period, usually a year. A rate which allows accurate comparison of investments is the Effective Annual Rate (EAR).
- EAR Definition: Interest rate annualized using compound rather than simple interest.

$$1 + EAR = [1 + r_f(T)]^n = [1 + r_f(T)]^{1/T} = [1 + T \times APR]^{1/T} \tag{5.8}$$

where $r_f(T)$ total returns over a holding period of T years (n then means periods per year). APR is annual percent rate (next slide).

Annual Percentage Rates

- APR: Annualized interest rate using simple rather than compound interest

$$APR = r_f(T) * n = \frac{r_f(T)}{T} = \frac{(1 + EAR)^T - 1}{T}$$

where n is number of periods per year, and T is the investment horizon (in years).

APR is usually used to report the rates on short-term investments (less than a year), however conceptually it applies to a period of any length.

Table 5.1 EAR versus APR

Compounding Period	T	$EAR = [1 + r_f(T)]^{1/T} - 1 = .058$		$APR = r_f(T)*(1/T) = .058$	
		$r_f(T)$	$APR = [(1 + EAR)^T - 1]/T$	$r_f(T)$	$EAR = (1 + APR*T)^{(1/T)} - 1$
1 year	1.0000	.0580	.05800	.0580	.05800
6 months	0.5000	.0286	.05718	.0290	.05884
1 quarter	0.2500	.0142	.05678	.0145	.05927
1 month	0.0833	.0047	.05651	.0048	.05957
1 week	0.0192	.0011	.05641	.0011	.05968
1 day	0.0027	.0002	.05638	.0002	.05971
Continuous			$r_{cc} = \ln(1 + EAR) = .05638$		$EAR = \exp(r_{cc}) - 1 = .05971$

Table 5.1

Annual percentage rates (APR) and effective annual rates (EAR). In the first set of columns, we hold the equivalent annual rate (EAR) fixed at 5.8% and find APR for each holding period. In the second set of columns, we hold APR fixed at 5.8% and solve for EAR.

excel
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Spreadsheet 5.1 Walkthrough video

Continuous compounding

- The difference between APR and EAR grows with the frequency of compounding
- As T in $[1 + T \times APR]^{1/T}$ approaches zero we approach continuous compounding

$$
\begin{aligned}
1 + EAR &= \exp(r_{cc}) = e^{r_{cc}} \\
r_{cc} &= \ln(1 + EAR)
\end{aligned}
\tag{5.9}
$$

Concept check

CONCEPT CHECK 5.2

A bank offers two alternative interest schedules for a savings account of $100,000 locked in for 3 years: (*a*) a monthly rate of 1%; (*b*) an annually, continuously compounded rate (r_{cc}) of 12%. Which alternative should you choose?

Risk and Risk Premiums

- Rates of Return: Single Period

$$HPR = \frac{P_1 - P_0 + D_1}{P_0} \tag{5.10}$$

where

$$\begin{aligned} HPR &= \text{Holding period return} \\ P_0 &= \text{Beginning price} \\ P_1 &= \text{Ending price} \\ D_1 &= \text{Dividend during period one} \end{aligned}$$

Rates of Return: Single Period Example

$$
\begin{aligned}
\text{Ending Price} &= \$110 \\
\text{Beginning Price} &= \$100 \\
\text{Dividend} &= \$4
\end{aligned}
$$

$$HPR = \frac{\$110 - \$100 + \$4}{\$100} = 0.14\text{, or } 14\%$$

Expected Return and Standard Deviation

Future returns are random variables—one value is realized out of many possible, depending on the random state of the economy (out of many possible states) which occurs. Each possible state has a probability of happening.

- Expected Return–probability-weighted average of all possible future returns.

$$E(r) = \sum_s p(s)r(s) \tag{5.11}$$

where

$$p(s) = \text{Probability of a state}$$
$$r(s) = \text{Return if state occurs}$$
$$s = \text{State, e.g. boom, bust, or normal}$$

Expected Return and Standard Deviation

- Variance
  - is the expectation of the squared deviation of a random variable from its mean; measures how far a set of (random) numbers are spread out from their mean[2].

$$\sigma^2 = \sum_s p(s)\,[r(s) - E(r)]^2 \tag{5.12}$$

- Standard deviation

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_s p(s)\,[r(s) - E(r)]^2}$$

[2] source: Wikipedia

Spreadsheet 5.1 Expected Returns and Standard Deviations

	A	B	C	D	E	F	G	H	I
1									
2									
3	Purchase Price =		$100			T-bill Rate =	0.04		
4									
5							Squared		Squared
6	State of the		Year-End	Cash		Deviations	Deviations	Excess	Deviations
7	Market	Probability	Price	Dividends	HPR	from Mean	from Mean	Returns	from Mean
8	Excellent	0.25	126.50	4.50	0.3100	0.2124	0.0451	0.2700	0.0451
9	Good	0.45	110.00	4.00	0.1400	0.0424	0.0018	0.1000	0.0018
10	Poor	0.25	89.75	3.50	−0.0675	−0.1651	0.0273	−0.1075	0.0273
11	Crash	0.05	46.00	2.00	−0.5200	−0.6176	0.3815	−0.5600	0.3815
12	Expected Value (mean)	SUMPRODUCT(B8:B11, E8:E11) =			0.0976				
13	Variance of HPR			SUMPRODUCT(B8:B11, G8:G11) =			0.0380		
14	Standard Deviation of HPR					SQRT(G13) =	0.1949		
15	Risk Premium					SUMPRODUCT(B8:B11, H8:H11) =		0.0576	
16	Standard Deviation of Excess Return						SQRT(SUMPRODUCT(B8:B11, I8:I11)) =		0.1949

Spreadsheet 5.1

$E(r)$ and σ, Scenario Example

- From Spreadsheet 5.1, column HPR

$$E(r) = (.25)(.31) + (.45)(.14) + (.25)(-.0675) + (0.05)(-0.52)$$
$$E(r) = .0976\text{, or } 9.76\%$$

- The variance is computed as

$$\begin{aligned}\sigma^2 =& 25(.31 - 0.0976)^2 + .45(.14 - .0976)^2 \\ &+ .25(-0.0675 - 0.0976)^2 + .05(-.52 - .0976)^2 \\ =& .038\end{aligned}$$

- Standard deviation is:

$$\sigma = \sqrt{\sigma^2} = \sqrt{0.038} = 0.1949 = 19.49\%$$

Concept check

CONCEPT CHECK 5.3

You invest $27,000 in a corporate bond selling for $900 per $1,000 par value. Over the coming year, the bond will pay interest of $75 per $1,000 of par value. The price of the bond at year's end will depend on the level of interest rates that will prevail at that time. You construct the following scenario analysis:

Interest Rates	Probability	Year-End Bond Price
High	.2	$850
Unchanged	.5	915
Low	.3	985

Your alternative investment is a T-bill that yields a sure rate of return of 5%. Calculate the HPR for each scenario, the expected rate of return, and the risk premium on your investment. What is the expected end-of-year dollar value of your investment?

Problem 5-7

Suppose your expectations regarding the stock price are as follows:

State of the Market	Probability	Ending Price	HPR (including dividends)
Boom	.35	$140	44.5%
Normal growth	.30	110	14.0
Recession	.35	80	−16.5

Compute the mean and standard deviation of the HPR on stocks. You can use the excel template: click here

Time Series Analysis of Past Returns

- True means and variances are unobservable because we don't actually know possible scenarios like the one in the examples
- So we must estimate the means and variances. Estimates contain errors in them. Question: what is the difference between error in the estimate and bias in the estimate?

Arithmetic and Geometric Averages

- Arithmetic Average

$$E(r) = \sum_{s=1}^{n} p(s)r(s) = \frac{1}{n}\sum_{s=1}^{n} r(s) \tag{5.13}$$

- Geometric average

$$\begin{aligned} \textit{Terminal value} &= (1 + r_1) \times (1 + r_2) \times \cdots \times (1 + r_n) \\ (1 + g)^n &= \textit{Terminal value} \\ g &= \textit{Terminal value}^{1/n} - 1 \end{aligned} \tag{5.14}$$

  - Practitioners call g time-weighted (as opposed to dollar-weighted) average return
  - The larger the swings in rates of return, the greater the discrepancy between

Spreadsheet 5.2 Arithmetic and Geometric Average Example

	A	B	C	D	E	F
1						
2						
3		Implicitly Assumed		Squared	Gross HPR =	Wealth
4	Period	Probability = 1/5	HPR (decimal)	Deviation	1 + HPR	Index*
5	2001	.2	−0.1189	0.0196	0.8811	0.8811
6	2002	.2	−0.2210	0.0586	0.7790	0.6864
7	2003	.2	0.2869	0.0707	1.2869	0.8833
8	2004	.2	0.1088	0.0077	1.1088	0.9794
9	2005	.2	0.0491	0.0008	1.0491	1.0275
10	Arithmetic average	AVERAGE(C5:C9) =	0.0210			
11	Expected HPR	SUMPRODUCT(B5:B9, C5:C9) =	0.0210			
12		Standard deviation	SUMPRODUCT(B5:B9, D5:D9)^.5 =	0.1774		Check:
13			STDEV(C5:C9) =	0.1983		1.0054^5=
14			Geometric average return	GEOMEAN(E5:E9) – 1 =	0.0054	1.0275
15	*The value of $1 invested at the beginning of the sample period (1/1/2001).					

[Spreadsheet 5.2]

Concept check

CONCEPT CHECK 5.4

You invest $1 million at the beginning of 2018 in an S&P 500 stock-index fund. Given the rate of return for 2018, –40%, what rate of return in 2019 will be necessary for your portfolio to recover to its original value?

Estimating Variance and Standard Deviation

- Estimated Variance - modifying eq. (5.12) for using historical data:
  - Expected value of squared deviations

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{s=1}^{n}[r(s) - \bar{r}]^2 \quad (5.16)$$

  - $\hat{\sigma}^2$ is biased because $\bar{r}$ is an estimate of $E(r)$, i.e. has error
    click here for details

- Unbiased estimated standard deviation

$$\hat{\sigma}^2 = \frac{n}{n-1} \times \frac{1}{n}\sum_{s=1}^{n}[r(s) - \bar{r}]^2$$

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{s=1}^{n}[r(s) - \bar{r}]^2} \quad (5.17)$$

Mean and SD Estimates from Higher-Frequency Observations

- Observation frequency has no impact on the accuracy of mean estimates.
- It is the duration of a sample time series (as opposed to the number of observations) that improves accuracy.
- Hence use the longest sample to estimate mean returns.
- In contrast to the mean, the accuracy of estimates of the standard deviation can be made more precise by increasing the number of observations.
- The T-month variance is T times the 1-month variance.

The Reward-to-Volatility (Sharpe) Ratio

- Excess Return
  - The difference in any particular period between the actual rate of return on a risky asset and the actual risk-free rate
- Risk Premium
  - The difference between the expected HPR on a risky asset and the risk-free rate
- Sharpe Ratio

$$\text{Sharpe Ratio} = \frac{\text{Risk premium}}{\text{SD of excess return}} \qquad (5.18)$$

Concept check

CONCEPT CHECK 5.5

Using the annual returns for years 2003–2005 in Spreadsheet 5.2,

a. Compute the arithmetic average return.

b. Compute the geometric average return.

c. Compute the standard deviation of returns.

d. Compute the Sharpe ratio assuming the risk-free rate was 6% per year.

Spreadsheet 5.2

The Normal Distribution

- Investment management is easier when returns are normal
  - Standard deviation is a good measure of risk when returns are symmetric
  - If security returns are symmetric, portfolio returns will be as well
  - Future scenarios can be estimated using only the mean and the standard deviation
  - The dependence of returns across securities can be summarized using only the pairwise correlation coefficients

Figure 5.4 The Normal Distribution

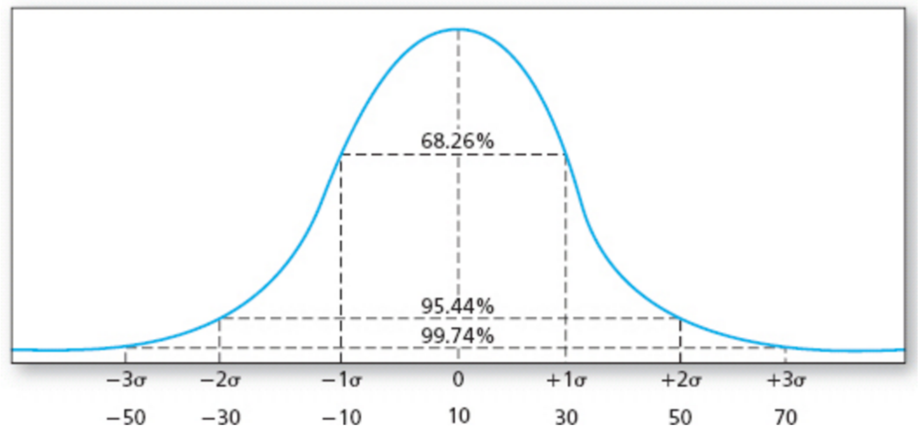


Figure 5.4 The normal distribution with mean 10% and standard deviation 20%.

Example 5.10 Using the Normal Distribution

- Suppose the monthly rate of return on the S&P 500 is approximately normally distributed with a mean of 1% and standard deviation of 6%. What is the probability that the return on the index in any month will be negative? (use the Excel function NORMDIST)

CONCEPT CHECK 5.6

What is the probability that the return on the index in Example 5.10 will be below –15%?

Normality and Risk Measures

- What if excess returns are not normally distributed?
  - Standard deviation is no longer a complete measure of risk
  - Sharpe ratio is not a complete measure of portfolio performance
  - Need to consider skewness and kurtosis

$$\text{Skewness} = \mathit{Average}\left[\frac{(R-\bar{R})^3}{\hat{\sigma}^3}\right] \tag{5.19}$$

$$\text{Kurtosis} = \mathit{Average}\left[\frac{(R-\bar{R})^4}{\hat{\sigma}^4}\right] \tag{5.20}$$

Figure 5.5A Normal and Skewed Distributions

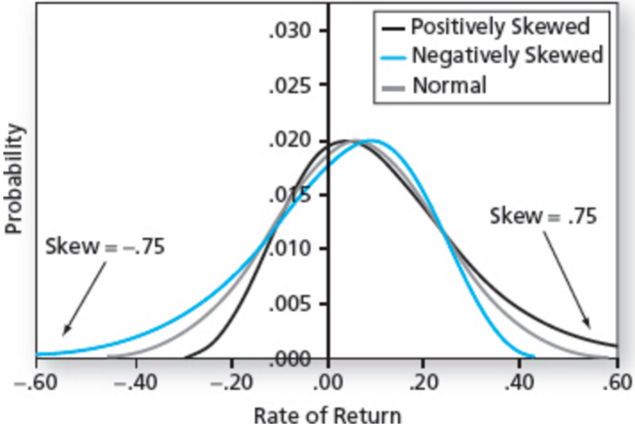


Figure 5.5A Normal and skewed distributions (mean = 6%, SD = 17%)

Figure 5.5B Normal and Fat-Tailed Distributions

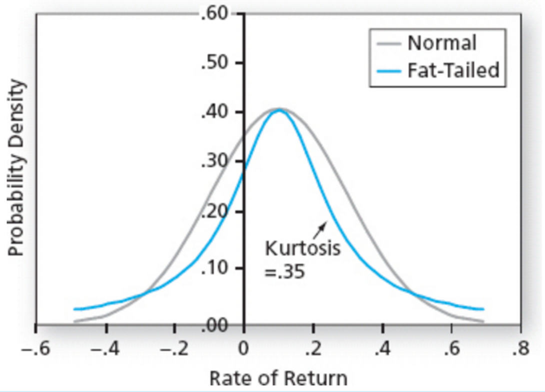


Figure 5.5B Normal and fat-tailed distributions (mean = .1, SD = .2)

Concept check 5.7: Estimate the skew and kurtosis of the five rates in Spreadsheet 5.2

Normality and Risk Measures

- Value at Risk (VaR)
  - Loss corresponding to a very low percentile of the entire return distribution, such as the fifth or first percentile return
- Expected Shortfall (ES)
  - Also called conditional tail expectation (CTE), focuses on the expected loss in the worst-case scenario (left tail of the distribution)
  - More conservative measure of downside risk than VaR

Normality and Risk Measures

- Lower Partial Standard Deviation (LPSD)
  - and the Sortino Ratio
  - Similar to usual standard deviation, but uses only negative deviations from the risk-free return, thus, addressing the asymmetry in returns issue
  - Sortino Ratio (replaces Sharpe Ratio)
    - The ratio of average excess returns to LPSD

Historic Returns on Risk Portfolios

- The second half of the 20th century, politically and economically the most stable sub-period, offered the highest average returns
- Firm capitalization is highly skewed to the right: Many small but a few gigantic firms
- Average realized returns have generally been higher for stocks of small rather than large capitalization firms

Table 5.3 Portfolio statistics

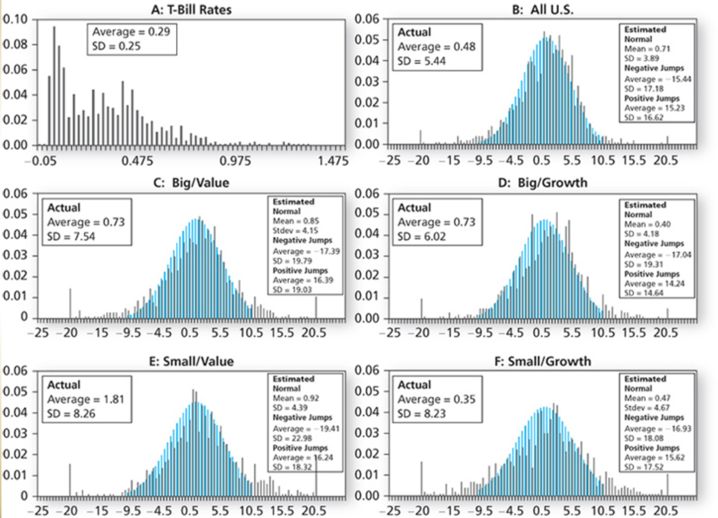
	All U.S.[a]	Big/Value	Big/Growth	Small/Value	Small/Growth
July 1926					
Number of firms	427	37	85	90	43
Average capitalization ($ mil)	57	39	108	5	8
Average B/M ratio[b]	1.02	2.36	0.45	3.6	0.81
January 1950					
Number of firms	899	73	196	197	75
Average capitalization ($ mil)	69	77	186	7	11
Average B/M ratio[b]	1.18	2.60	0.50	2.95	0.67
January 2000					
Number of firms	5,495	150	576	1,709	1,158
Average capitalization ($ mil)	2,545	3,542	18,246	106	299
Average B/M ratio[b]	0.52	1.38	0.14	1.70	0.22
September 2012					
Number of firms	3,383	153	408	1,065	672
Average capitalization ($ mil)	4,470	13,325	18,070	297	582
Average B/M ratio[b]	0.68	1.32	0.25	1.33	0.26

Table 5.3

Number of firms, average capitalization, and average B/M ratios of portfolios

Notes: [a] Value weighted, hence dominated by big stocks
[b] B/M ratio are sampled in midyears
Source: Professor Kenneth French's Web site, **http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.**

Figure 5.6


A: T-Bill Rates
Average = 0.29
SD = 0.25
0.10 0.08 0.06 0.04 0.02 0.00
−0.05 0.475 0.975 1.475
B: All U.S.
Actual
Average = 0.48
SD = 5.44
Estimated
Normal
Mean = 0.71
SD = 3.89
Negative Jumps
Average = −15.44
SD = 17.18
Positive Jumps
Average = 15.23
SD = 16.62
0.06 0.05 0.04 0.03 0.02 0.01 0.00
−25 −20 −15 −9.5 −4.5 0.5 5.5 10.5 15.5 20.5
C: Big/Value
Actual
Average = 0.73
SD = 7.54
Estimated
Normal
Mean = 0.85
Stdev = 4.15
Negative Jumps
Average = −17.39
SD = 19.79
Positive Jumps
Average = 16.39
SD = 19.03
0.06 0.05 0.04 0.03 0.02 0.01 0
−25 −20 −15 −9.5 −4.5 0.5 5.5 10.5 15.5 20.5
D: Big/Growth
Actual
Average = 0.73
SD = 6.02
Estimated
Normal
Mean = 0.40
SD = 4.18
Negative Jumps
Average = −17.04
SD = 19.31
Positive Jumps
Average = 14.24
SD = 14.64
0.06 0.05 0.04 0.03 0.02 0.01 0.00
−25 −20 −15 −9.5 −4.5 0.5 5.5 10.5 15.5 20.5
E: Small/Value
Actual
Average = 1.81
SD = 8.26
Estimated
Normal
Mean = 0.92
SD = 4.39
Negative Jumps
Average = −19.41
SD = 22.98
Positive Jumps
Average = 16.24
SD = 18.32
0.06 0.05 0.04 0.03 0.02 0.01 0
−25 −20 −15 −9.5 −4.5 0.5 5.5 10.5 15.5 20.5
F: Small/Growth
Actual
Average = 0.35
SD = 8.23
Estimated
Normal
Mean = 0.47
Stdev = 4.67
Negative Jumps
Average = −16.93
SD = 18.08
Positive Jumps
Average = 15.62
SD = 17.52
0.06 0.05 0.04 0.03 0.02 0.01 0.00
−25 −20 −15 −9.5 −4.5 0.5 5.5 10.5 15.5 20.5

Historic Returns on Risk Portfolios

- Normal distribution is generally a good approximation of portfolio returns
  - VaR indicates no greater tail risk than is characteristic of the equivalent normal
  - The ES does not exceed 0.41 of the monthly SD, presenting no evidence against the normality
- However
  - Negative skew is present in some of the portfolios some of the time, and positive kurtosis is present in all portfolios all the time

Table 5.4a Average Excess Returns

- The second half of the 20th century, politically and economically the most stable subperiod, offered the highest average returns, particularly for the equally-weighted portfolios.

	All U.S.	Big/ Value	Big/ Growth	Small/ Value	Small/ Growth	Average of Four Comparison Portfolios
All years	7.52	12.34	10.98	26.28	8.38	14.49
21st century	1.82	8.80	14.51	17.89	4.83	11.51
20th cent. 2nd half	8.64	16.02	11.49	50.48	12.81	22.70
20th cent. 2nd quarter	8.44	11.50	9.83	17.05	7.20	11.40

Table 5.4A

Average excess returns over time

Table 5.4b Standard Deviations

- The second quarter of the 20th century, dominated by the Great Depression and legendary for upheaval in stock values, exhibits the highest standard deviations.

	All U.S.	Big/ Value	Big/ Growth	Small/ Value	Small/ Growth	Average of Four Comparison Portfolios
All years	20.46	29.25	20.79	41.41	32.80	31.06
21st century	20.08	24.08	20.93	28.93	29.49	25.86
20th cent. 2nd half	14.99	17.21	16.51	21.41	25.60	20.18
20th cent. 2nd quarter	28.72	46.59	27.61	63.74	45.08	45.76

Table 5.4B

Standard deviations over time

Table 5.4c Sharpe Ratios

- All portfolios attained their highest Sharpe ratios over the second half of the 20th century. The 21st century has witnessed the lowest performance from the large cap-weighted All U.S. portfolio

	All U.S.	Big/ Value	Big/ Growth	Small/ Value	Small/ Growth	Average of Four Comparison Portfolios
All years	0.37	0.42	0.53	0.63	0.26	0.46
21st century	0.09	0.37	0.69	0.62	0.16	0.46
20th cent. 2nd half	0.56	0.67	0.60	0.80	0.28	0.59
20th cent. 2nd quarter	0.30	0.34	0.42	0.79	0.28	0.46

Table 5.4C

Sharpe ratios over time

Table 5.4d VaR

- For the overall period, VaR indicates no greater tail risk than is characteristic of the equivalent normal. The worst excess VaR compared to the normal (−1.71% for the Big/Value portfolio in the 21st century) is less than a third of the monthly SD of this portfolio, 6.01%. Hence, VaR figures indicate that the normal is a decent approximation to the actual return distribution.

	All U.S.	Big/ Value	Big/ Growth	Small/ Value	Small/ Growth	Average of Four Comparison Portfolios
All years	0.12	0.93	0.85	2.81	0.72	1.33
21st century	−1.02	−1.71	0.34	−0.67	−0.37	−0.60
20th cent. 2nd quarter	0.06	−0.19	0.19	0.47	0.34	0.20
20th cent. 2nd half	−1.16	−0.09	−0.08	−0.39	0.18	−0.09

Table 5.4D

VaR (excess over normal distribution) expressed as a fraction of monthly standard deviation

Table 5.4e, Expected Shortfall (excess of normal distribution)

- While the most negative 5% of the actual observations are always worse than the equivalent normal, the differences are not substantial; we don't see evidence that seriously undermines the adequacy of the normality assumption.

	All U.S.	Big/ Value	Big/ Growth	Small/ Value	Small/ Growth	Average of Four Comparison Portfolios
All years	−0.41	−0.33	−0.34	−0.08	−0.25	−0.25
21st century	−0.28	−0.47	−0.17	−0.19	−0.11	−0.24
20th cent. 2nd quarter	−0.18	−0.06	−0.13	−0.07	−0.18	−0.11
20th cent. 2nd half	−0.77	−0.59	−0.77	−0.37	−0.26	−0.50

Table 5.4E

Expected shortfall (excess over normal distribution) expressed as a fraction of monthly standard deviation

Table 5.4f, Three-sigma Events

- There are from 2.9 to 9.7 3-sigma returns per 1,000 months, compared with only 0.6 to 1.0 for equivalent normal distributions; one excess jump is observed every 9 to 36 years; excess returns of 16 to 104 standard deviations. Still, jump risk does not appear large enough to affect the risk and return of long-term stock returns, and the normal distribution still works well.

	All U.S.	Big/ Value	Big/ Growth	Small/ Value	Small/ Growth
Number of negative jumps	7.7	4.8	9.7	2.9	3.9
Expected jumps for equivalent normal	0.9	0.9	0.8	0.6	1.0
Difference = excess jumps	6.8	4.0	8.9	2.3	2.8
Average years between excess jumps	12.24	21.06	9.42	36.23	29.37
Expected excess returns between extra jumps (in units of SD)	16.90	32.91	17.16	104.23	28.98

Table 5.4F

Incidence of negative 3-sigma returns

Table 5.4g Standard Deviation of Extreme Returns

- When we look at all excess returns below −10% in our history of 1,035 months, what is the SD of all these (extremely bad) returns?
- What would be the tail SD of a normal return with the same mean and overall SD as our sample, conditional on return falling below −10%?

	All U.S.	Big/Value	Big/Growth	Small/Value	Small/Growth
From 1,035-month history	17.18	19.79	19.31	22.98	18.08
From an equivalent normal	12.82	14.95	13.47	16.85	15.16
% Difference	33.99	32.35	43.35	36.39	19.23

Table 5.4G

Standard deviation conditional on excess return less than −10%

Figure 5.7 Returns Around the World

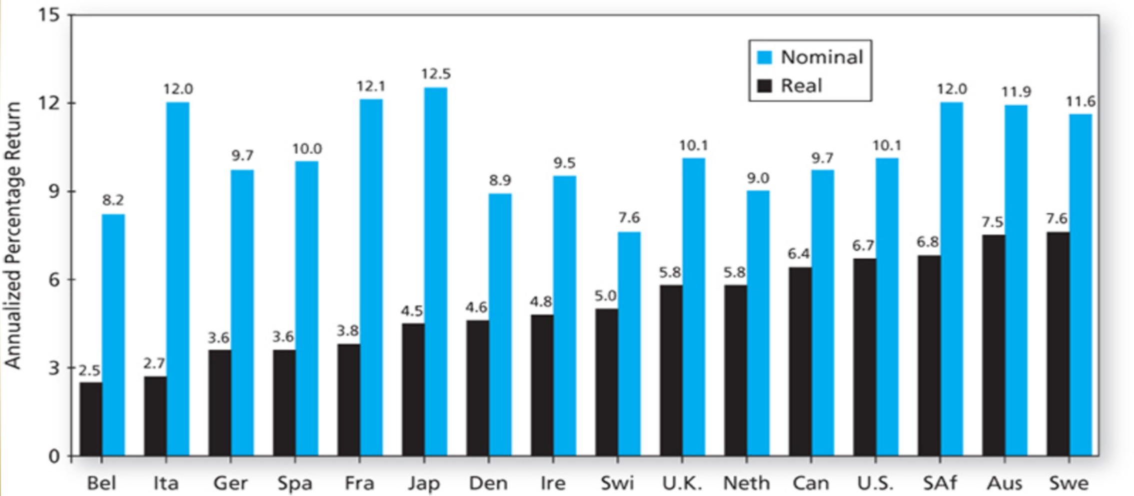
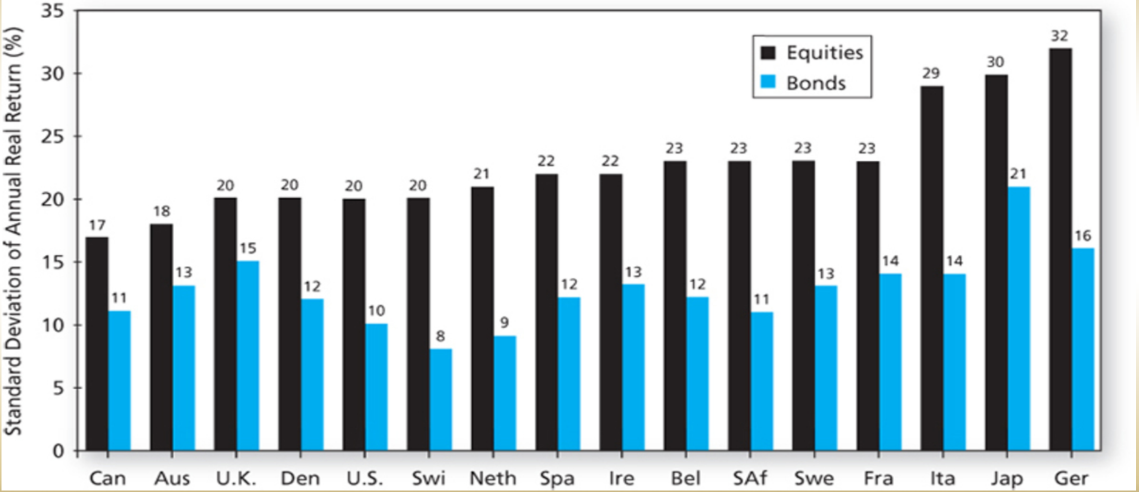


Figure 5.8 SD of Real Equity and Bond Portfolios



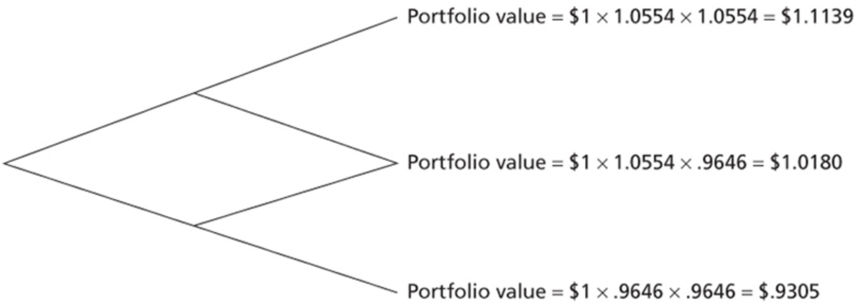
Long-Term Investments

The Power of Compounding

- Consider an investment of $1 over 25 years (300 months).
  - in a risky asset at a monthly return of 1% (12% per annum): $(1 + 0.01)^{300} = \$19.79$
  - in Treasuries at a monthly return of .5% (6% per annum): $(1 + 0.005)^{300} = \$4.46$
  - risk premium $= 19.79 - 4.46 = \$15.33$
- Why invest in Treasuries? Issue of risk-return tradeoffs.
- The risk of an investment that compounds at fluctuating rates over the long run is important, but is widely misunderstood.

Binomial Tree with Differing Rates

- Suppose each month it is equally likely that the rate of return is either 5.54% or −3.54%. The expected return is 1% ($= 0.5 \times 5.54 + 0.5 \times (-3.54)$), with a monthly st. dev. $\sigma_{mo} = \sqrt{0.5 \times (5.54 - 1)^2 + 0.5 \times (-3.54 - 1)^2} = 4.54\%$
- after two months the tree looks like this:



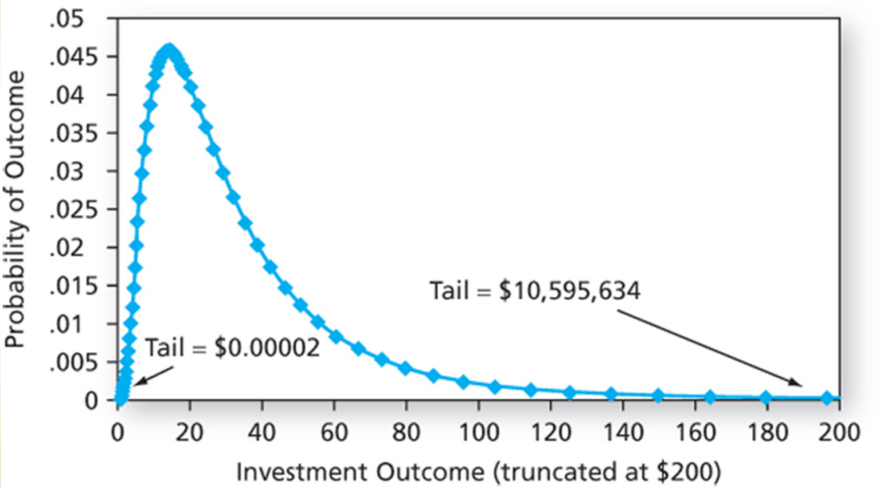
Terminal Value with Continuous Compounding

- “Growing” the binomial tree for 300 months will result in 301 different possible outcomes, whose distribution (is not normal and) looks like this:
- The probability of each of the 301 outcome can be obtained from Excel’s BINOMDIST function.
- This is a lognormal distribution due to the compounding. The sum of normally distributed random variables is also normally distributed but the product is not.
- When the continuously compounded rate of return on an asset is normally distributed, the effective rate of return will be lognormally distributed

Risk and the asymmetric lognormal distribution

- From the 301 possible outcomes and associated probabilities we compute the mean (\$19.79) and the standard deviation (\$18.09) of the terminal value.
- Can we use this standard deviation as a measure of risk even though the outcomes are very asymmetric (see next slide)?

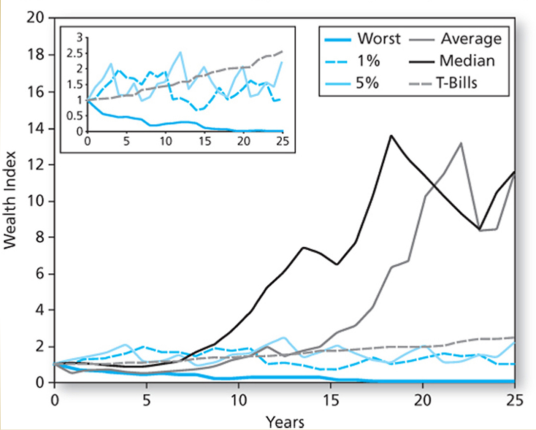
Figure 5.9 Probability of Investment with a Lognormal Distribution



Rules When Price is Lognormal

- When a normal rate, *g*, compounds by random shocks from instant to instant, the fluctuations do not produce symmetric effects on price.
  - A positive uptick raises the base, so the next tick is expected to be larger than the previous one. The reverse is true after a downtick
  - the base is smaller and the next tick is expected to be smaller.
- As a result, a sequence of positive shocks will have a larger upward effect than the downward effect of a sequence of negative shocks.
  - Thus, an upward drift is created just by volatility, even if *g* is zero.
  - St. Dev. rises only in proportion to the square root of time: $\sigma\sqrt{(T)}$.
  - This appears to offer a mitigation of investment risk in the long run.
  - In addition, the probability of shortfall is smaller in the long-run (see Ex. 5.11)
  - However, the magnitude of losses and their timing (losing when you most need the money) will make portfolio insurance surprisingly costly in the long run.

Figure 5.12. Possible wealth trajectories and shortfall



- The bottom 5% still results in a significant shortfall relative to the T-bill portfolio. The analysis rejects the notion that stocks become less risky in the long run.

Problem 5-18

Consider these long-term investment data:

- The price of a 10-year $100 par zero coupon inflation-indexed bond is $84.49.
- A real-estate property is expected to yield 2% per quarter (nominal) with a SD of the (effective) quarterly rate of 10%.

  a. Compute the annual rate on the real bond.

  b. Compute the CC annual risk premium on the real-estate investment.

  c. Use the appropriate formula and Excel Solver or Goal Seek to find the SD of the CC annual excess return on the real-estate investment.

  d. What is the probability of loss or shortfall after 10 years?

Tip: Reviewing Example 5.11 will help you solve this problem.