Chapter 6, Capital Allocation To Risky Assets

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Chapter Overview

Risk and Risk Aversion

- Speculation
  - Taking considerable risk for a commensurate gain
    - Parties have heterogeneous expectations
    - “considerable risk”—risk is sufficient to affect the decision.
    - “commensurate gain” = a positive risk premium
- Gamble
  - Bet on an uncertain outcome for enjoyment
    - Parties assign the same probabilities to the possible outcomes
    - “fair game”—risky investment with a zero risk premium

Concept Check

CONCEPT CHECK 6.1

Assume that dollar-denominated T-bills in the United States and pound-denominated bills in the United Kingdom offer equal yields to maturity. Both are short-term assets, and both are free of default risk. Neither offers investors a risk premium. However, a U.S. investor who holds U.K. bills is subject to exchange rate risk, because the pounds earned on the U.K. bills eventually will be exchanged for dollars at the future exchange rate. Is the U.S. investor engaging in speculation or gambling?

Utility Values

- Investors are willing to consider:
  - Risk-free assets
  - Speculative positions with positive risk premiums
- Portfolio attractiveness increases with expected return and decreases with risk
- What happens when return increases with risk?

Table 6.1 Available Risky Portfolios

Table 6.1

Available risky portfolios (Risk-free rate = 5%)

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	7%	5%
M (medium risk)	4	9	10
H (high risk)	8	13	20

- Each portfolio receives a utility score to assess the investor's risk/return trade off

Risk Aversion and Utility Values

- Utility Function

$$U = \text{Utility}$$
$$E(r) = \text{Expected return on the asset or portfolio}$$
$$A = \text{Coefficient of risk aversion}$$
$$\sigma^2 = \text{Variance of returns}$$
$$^1/_2 = \text{A scaling factor}$$

$$U = E(r) - {}^1/_2 A\sigma^2 \tag{6.1}$$

Table 6.2 Utility Scores

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r) = .07$; $\sigma = .05$]	Utility Score of Portfolio M [$E(r) = .09$; $\sigma = .10$]	Utility Score of Portfolio H [$E(r) = .13$; $\sigma = .20$]
2.0	$.07 - ½ \times 2 \times .05^2 = .0675$	$.09 - ½ \times 2 \times .1^2 = .0800$	**$.13 - ½ \times 2 \times .2^2 = .09$**
3.5	$.07 - ½ \times 3.5 \times .05^2 = .0656$	**$.09 - ½ \times 3.5 \times .1^2 = .0725$**	$.13 - ½ \times 3.5 \times .2^2 = .06$
5.0	$.07 - ½ \times 5 \times .05^2 = .0638$	**$.09 - ½ \times 5 \times .1^2 = .0650$**	$.13 - ½ \times 5 \times .2^2 = .03$

Table 6.2

Utility scores of alternative portfolios for investors with varying degrees of risk aversion

Concept Check

CONCEPT CHECK 6.2

A portfolio has an expected rate of return of 20% and standard deviation of 30%. T-bills offer a safe rate of return of 7%. Would an investor with risk-aversion parameter $A = 4$ prefer to invest in T-bills or the risky portfolio? What if $A = 2$?

Estimating Risk Aversion

- Use questionnaires
- Observe individuals' decisions when confronted with risk
- Observe how much people are willing to pay to avoid risk

Estimating Risk Aversion

- Mean-Variance (M-V) criterion
  - Portfolio A dominates portfolio B if:

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

Figure 6.1 Mean-Variance Criterion

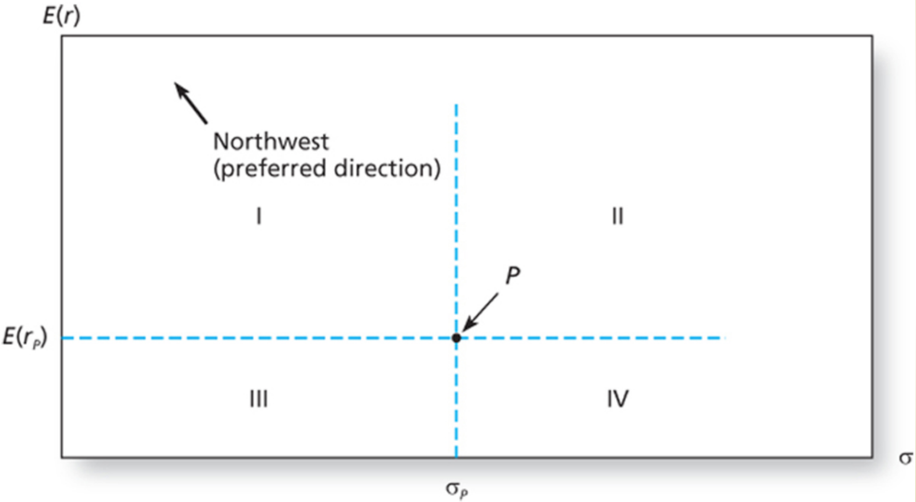
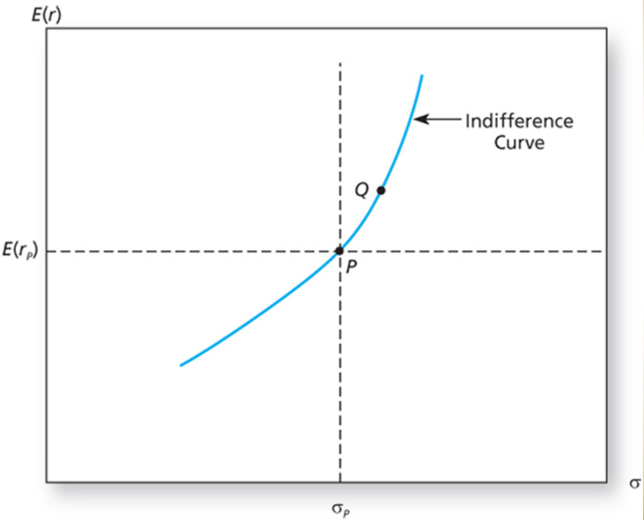


Figure 6.2 Indifference Curves



Concept Check

CONCEPT CHECK 6.3

a. How will the indifference curve of a less risk-averse investor compare to the indifference curve drawn in Figure 6.2?

b. Draw both indifference curves passing through point *P*.

Table 6.3 Indifference Curves for $A = 4$

Table 6.3

Utility values of possible portfolios for investor with risk aversion, $A = 4$

Expected Return, $E(r)$	Standard Deviation, σ	Utility = $E(r) - ½ A\sigma^2$
.10	.200	$.10 - .5 \times 4 \times .04 = .02$
.15	.255	$.15 - .5 \times 4 \times .065 = .02$
.20	.300	$.20 - .5 \times 4 \times .09 = .02$
.25	.339	$.25 - .5 \times 4 \times .115 = .02$

Capital Allocation Across Risky and Risk-Free Portfolios

- Asset Allocation
  - The choice among broad asset classes that represents a very important part of portfolio construction
- The simplest way to control risk is to manipulate the fraction of the portfolio invested in risk-free assets versus the portion invested in the risky assets

Basic Asset Allocation Example

Total market value	\$300,000
Risk-free money market fund	\$90,000
Equities	\$113,400
Bonds (long-term)	\$96,600
Total risk assets	\$210,000

$$E: \quad w_E = \frac{113,400}{210,000} = 0.54 \qquad B: \quad w_B = \frac{96,600}{210,000} = 0.46$$

Basic Asset Allocation Example

Let:

$$y = \text{Weight of the risky portfolio, P, in the complete portfolio}$$
$$(1-y) = \text{Weight of risk-free assets}$$

$$y = \frac{210,000}{300,000} = 0.70 \text{ (risky assets)} \quad (1-y) = \frac{90,000}{300,000} = 0.30 \text{ (risk-free asset)}$$

$$E: \quad \frac{113,400}{300,000} = 0.378 \qquad B: \quad \frac{96,600}{300,000} = 0.322$$

Concept Check

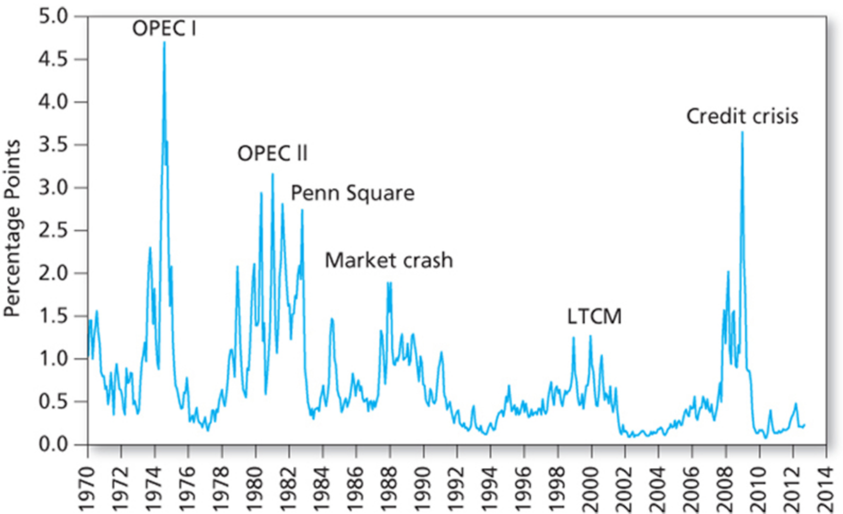
CONCEPT CHECK 6.4

What will be the dollar value of your position in equities (*E*), and its proportion in your overall portfolio, if you decide to hold 50% of your investment budget in Ready Asset?

The Risk-Free Asset

- Only the government can issue default-free securities
- A security is risk-free in real terms only if its price is indexed and maturity is equal to investor's holding period
- T-bills viewed as "the" risk-free asset
- Money market funds also considered risk-free in practice

Figure 6.3 Spread Between 3-Month CD and T-bill Rates



Portfolios of One Risky Asset and a Risk-Free Asset

- It's possible to create a complete portfolio by splitting investment funds between safe and risky assets

Let $y =$ Portion allocated to the risky portfolio, P

$(1-y) =$ Portion to be invested in risk-free asset, F

$$r_f = 7\% \qquad \sigma_{rf} = 0\%$$

$$E(r_P) = 15\% \qquad \sigma_P = 22\%$$

- The expected return on the complete portfolio

$$E(r_C) = yE(r_P) + (1-y)r_f = r_f + y[E(r_P) - r_f] = 7 + y(15-7) \tag{6.3}$$

- The risk of the complete portfolio is

$$\sigma_C = y\sigma_P = 22y \tag{6.4}$$

One Risky Asset and a Risk-Free Asset: Example

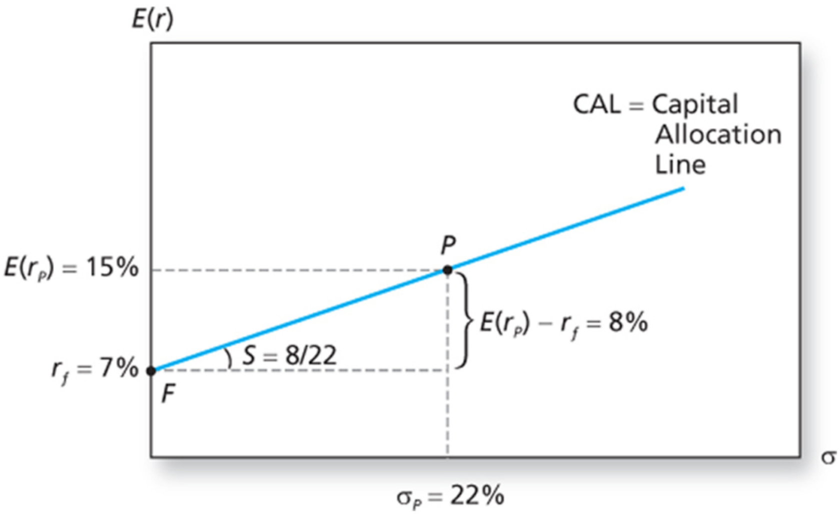
- Rearrange and substitute $y = \sigma_C / \sigma_P$

$$\begin{aligned} E(r_C) =& r_f + y[E(r_P - r_f] \\ =& r_f + \frac{\sigma_C}{\sigma_P}[E(r_P - r_f] = 7 + \frac{8}{22}\sigma_C \end{aligned} \tag{6.5}$$

- and the slope is:

$$S = \frac{E(r_P) - r_f}{\sigma_P} = \frac{8}{22} \tag{6.6}$$

Figure 6.4 The Investment Opportunity Set



Concept Check

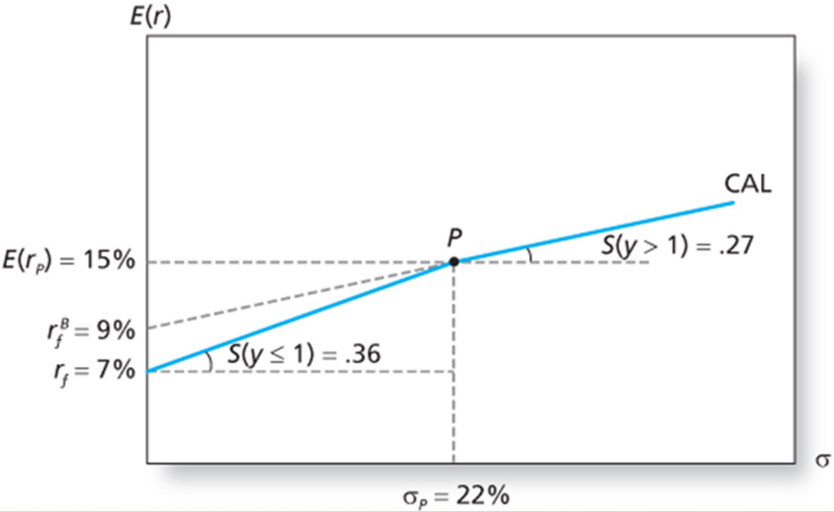
CONCEPT CHECK 6.5

Can the reward-to-volatility (Sharpe) ratio, $S = [E(r_C) - r_f]/\sigma_C$, of any combination of the risky asset and the risk-free asset be different from the ratio for the risky asset taken alone, $[E(r_P) - r_f]/\sigma_P$, which in this case is .36?

Risky–Risk-free Allocation with Leverage: Example

- Capital allocation line with leverage and differential borrowing rates
  - Lend at $r_f = 7\%$ and borrow at $r_f = 9\%$
    - Lending range slope $= 8/22 = 0.36$
    - Borrowing range slope $= 6/22 = 0.27$
  - CAL kinks at P

Figure 6.5 CAL with Leverage and Differential Rates



Concept Check

CONCEPT CHECK 6.6

Suppose that there is an upward shift in the expected rate of return on the risky asset, from 15% to 17%. If all other parameters remain unchanged, what will be the slope of the CAL for $y \leq 1$ and $y > 1$?

Risk Tolerance and Asset Allocation

- The investor must choose one optimal portfolio, C, from the set of feasible choices
- Expected return of the complete portfolio:

$$E(r_C) = r_f + y[E(r_P - r_f]$$

- Variance:

$$\sigma_C^2 = y^2 \sigma_P^2$$

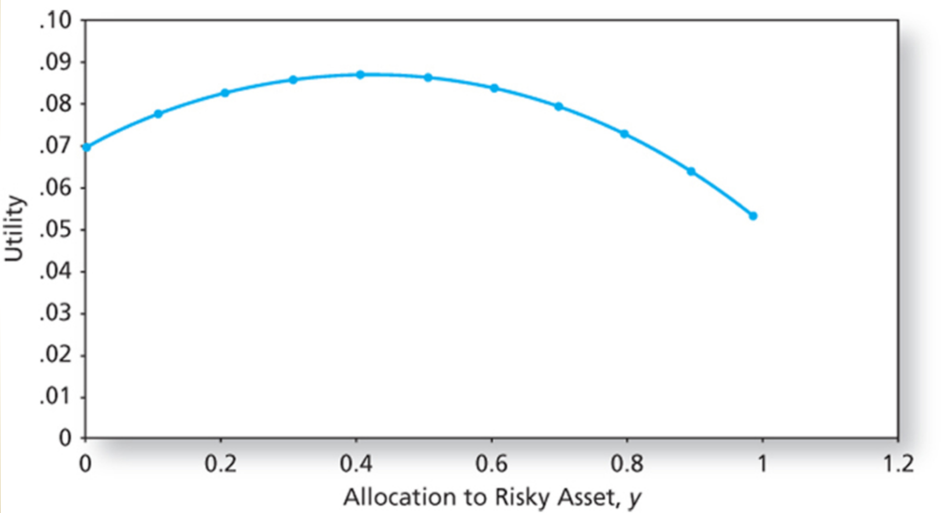
Table 6.4 Utility Levels for Different Positions

Table 6.4

Utility levels for various positions in risky assets (y) for an investor with risk aversion $A = 4$

(1) y	(2) $E(r_C)$	(3) σ_C	(4) $U = E(r) - ½A\sigma^2$
0	.070	0	.0700
0.1	.078	.022	.0770
0.2	.086	.044	.0821
0.3	.094	.066	.0853
0.4	.102	.088	.0865
0.5	.110	.110	.0858
0.6	.118	.132	.0832
0.7	.126	.154	.0786
0.8	.134	.176	.0720
0.9	.142	.198	.0636
1.0	.150	.220	.0532

Figure 6.6 Utility As Function of, *y*, the Allocation to *P*



Finding the Optimal Complete Portfolio

- To find the optimal position in the risky asset, solve:

$$\underset{y}{\text{Max}} \quad U = E(r_C) - {}^1\!/_2 A\sigma^2 = r_f + y[E(r_P) - r_f] - {}^1\!/_2 A y^2 \sigma_P^2$$

which yields the optimal weight y^*:

$$y^* = \frac{E(r_P) - r_f}{A\sigma_p^2} \tag{6.7}$$

Example Capital Allocation

- Using our numerical example [$rf = 7\%$, $E(r_P) = 15\%, \sigma_P = 22\%$], and expressing all returns as decimals, the optimal solution for an investor with a coefficient of risk aversion $A = 4$ is:

$$y^* = \frac{0.15 - 0.07}{4 \times 0.22^2} = 0.41$$

- With 41% invested in the risky portfolio, the expected return and standard deviation of the complete portfolio are

$$E(r_C) = 7 + [0.41 \times (15 - 7)] = 10.28\%$$
$$\sigma_C = 0.41 \times 22 = 9.02\%$$

- The risk premium of the complete portfolio is $E(r_C) r_f = 3.28\%$, which is obtained by taking on a portfolio with a standard deviation of 9.02%.

- Notice that $3.28/9.02 = 0.36$, which is the reward-to-volatility (Sharpe) ratio

Building an Indifference Curves

- Consider an investor with $A = 4$ who currently holds all her wealth in a risk-free portfolio: $r_f = 5\%$, $\sigma = 0$. From eq. (6.1), utility value is $U = 0.05$
- Now find $E(r)$ the investor would require to maintain the same level of utility when holding a risky portfolio, say, with $\sigma = 10\%$

$$U = E(r) - 0.5 \times A \times \sigma^2$$
$$0.05 = E(r) - 0.5 \times 4 \times 0.1^2$$

- This implies the necessary expected return increases to:

$$\begin{aligned} \text{Required } E(r) &= 0.05 + 0.5 \times A \times \sigma^2 \\ &= 0.05 + 0.5 \times 4 \times 0.1^2 = 0.07 \end{aligned} \tag{6.8}$$

Table 6.5 Calculations of Indifference Curves

Table 6.5

Spreadsheet calculations of indifference curves (Entries in columns 2–4 are expected returns necessary to provide specified utility value.)

	$A = 2$		$A = 4$	
σ	$U = .05$	$U = .09$	$U = .05$	$U = .09$
0	.0500	.0900	.050	.090
.05	.0525	.0925	.055	.095
.10	.0600	.1000	.070	.110
.15	.0725	.1125	.095	.135
.20	.0900	.1300	.130	.170
.25	.1125	.1525	.175	.215
.30	.1400	.1800	.230	.270
.35	.1725	.2125	.295	.335
.40	.2100	.2500	.370	.410
.45	.2525	.2925	.455	.495
.50	.3000	.3400	.550	.590

Figure 6.7 Indifference Curves for $U = .05$, $U = .09$, $A = 2$, $A = 4$

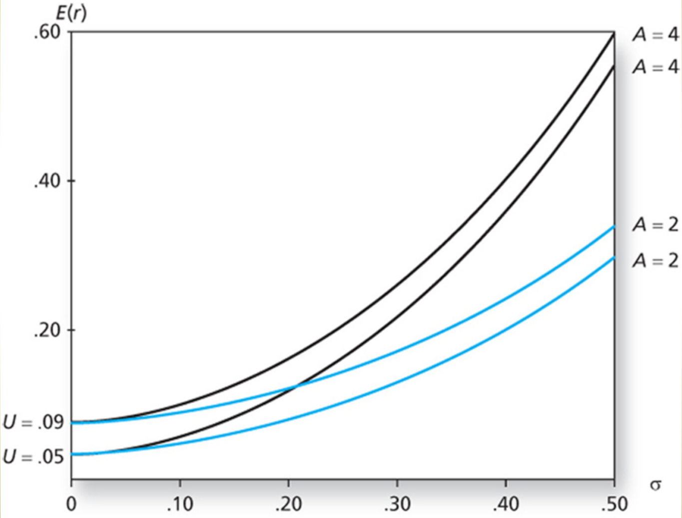


Figure 6.8 Finding the Optimal Complete Portfolio

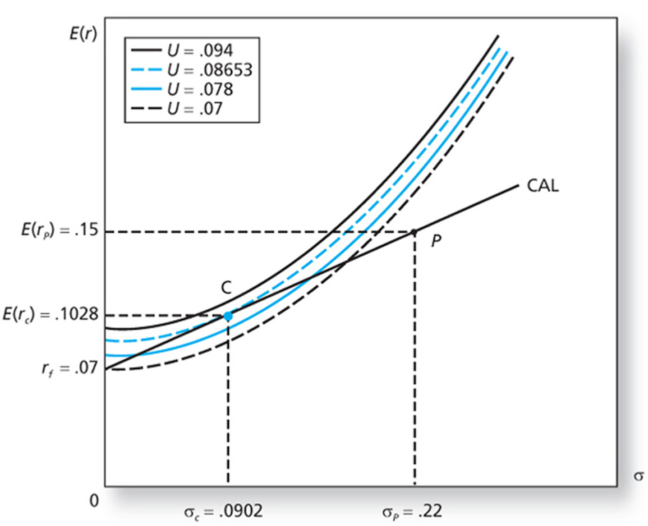


Table 6.6 Expected Returns on 4 IC and the CAL

Table 6.6

Expected returns on four indifference curves and the CAL. Investor's risk aversion is $A = 4$.

σ	$U = .07$	$U = .078$	$U = .08653$	$U = .094$	CAL
0	.0700	.0780	.0865	.0940	.0700
.02	.0708	.0788	.0873	.0948	.0773
.04	.0732	.0812	.0897	.0972	.0845
.06	.0772	.0852	.0937	.1012	.0918
.08	.0828	.0908	.0993	.1068	.0991
.0902	.0863	.0943	.1028	.1103	.1028
.10	.0900	.0980	.1065	.1140	.1064
.12	.0988	.1068	.1153	.1228	.1136
.14	.1092	.1172	.1257	.1332	.1209
.18	.1348	.1428	.1513	.1588	.1355
.22	.1668	.1748	.1833	.1908	.1500
.26	.2052	.2132	.2217	.2292	.1645
.30	.2500	.2580	.2665	.2740	.1791

Optimal portfolio highlights

- From Fig. 6.8 and Table 6.6, the indifference curve with U = .08653 is tangent to the CAL; the tangency point corresponds to the complete portfolio that maximizes utility.
- The tangency point occurs at $\sigma_C = 9.02\%$ and $E(r_C) = 10.28\%$, the risk-return parameters of the optimal complete portfolio with $y^* = 0.41$.
- The choice for y^* is determined by risk aversion (the slope of indifference curves) and the Sharpe ratio (the slope of the opportunity set).

Concept Check

CONCEPT CHECK 6.7

a. If an investor's coefficient of risk aversion is $A = 3$, how does the optimal asset mix change? What are the new values of $E(r_C)$ and σ_C?

b. Suppose that the borrowing rate, $r_f^B = 9\%$ is greater than the lending rate, $r_f = 7\%$. Show graphically how the optimal portfolio choice of some investors will be affected by the higher borrowing rate. Which investors will *not* be affected by the borrowing rate?

Passive Strategies: The Capital Market Line

- In the CAL, the determination of the assets to include in the risky portfolio, P, may result from a passive or an active strategy.
- The passive strategy avoids any direct or indirect security analysis
- Supply and demand forces may make such a strategy a reasonable choice for many investors
- A natural candidate for a passively held risky asset would be a well-diversified portfolio of common stocks such as the S&P 500

Passive Strategies: The Capital Market Line

- The Capital Market Line (CML)
  - Is a capital allocation line formed by investment in two passive portfolios:
    1. Virtually risk-free short-term T-bills (or a money market fund)
    2. Fund of common stocks that mimics a broad market index

Table 6.7

	Average Annual Returns		S&P 500 Portfolio			
Period	S&P 500 Portfolio	1-Month T-Bills	Risk Premium	Standard Deviation	Sharpe Ratio (Reward-to-Volatility)	Probability*
1926–2012	11.67	3.58	8.10	20.48	0.40	—
1989–2012	11.10	3.52	7.59	18.22	0.42	0.94
1968–1988	10.91	7.48	3.44	16.71	0.21	0.50
1947–1967	15.35	2.28	13.08	17.66	0.74	0.24
1926–1946	9.40	1.04	8.36	27.95	0.30	0.71

Table 6.7

Average annual return on large stocks and 1-month T-bills; standard deviation and Sharpe ratio of large stocks over time

*The probability that the estimate of the Sharpe ratio over 1926–2012 equals the true value and that we observe the reported, or an even more different Sharpe ratio for the subperiod.

Passive Strategies: The Capital Market Line

- From 1926 to 2012, the passive risky portfolio offered an average risk premium of 8.1% with a standard deviation of 20.48%, resulting in a reward-to-volatility ratio of .40
- Substituting these values in Eq. 6.7, we obtain:

$$y^* = \frac{E(r_M) - r_f}{A\sigma_M^2} = \frac{0.081}{A \times 0.2048^2} = 0.656$$

which implies a risk aversion of

$$A = \frac{0.081}{0.656 \times 0.2048^2} = 2.94$$

Concept Check

CONCEPT CHECK 6.8

Suppose that expectations about the S&P 500 index and the T-bill rate are the same as they were in 2012, but you find that a greater proportion is invested in T-bills today than in 2012. What can you conclude about the change in risk tolerance over the years since 2012?

St. Petersburg Paradox

- Pay fee to enter game. Then toss a coin until the first head appears and receive payoff: $R(n) = 2^n$, n = number of tails until first head is tossed.

Tails	Probability	Payoff = \$ $R(n)$	Probability × Payoff
0	1/2	\$1	\$1/2
1	1/4	\$2	\$1/2
2	1/8	\$4	\$1/2
3	1/16	\$8	\$1/2
⋮	⋮	⋮	⋮
n	$(1/2)^{n+1}$	\$$2^n$	\$1/2

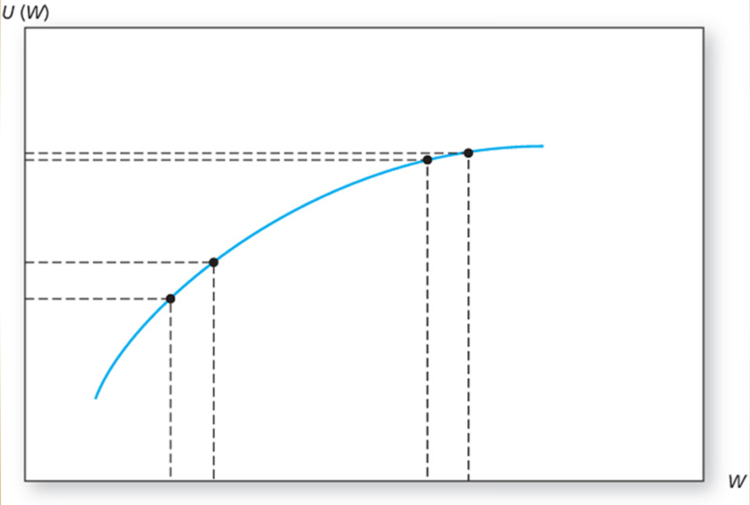
St. Petersburg Paradox

- The expected payoff is:

$$E(R) = \sum_{n=0}^{\infty} Pr(n)R(n) = \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

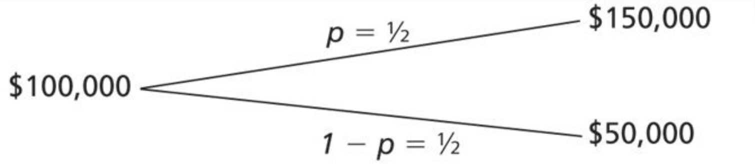
- Although the expected payoff is infinite, participants obviously will be willing to purchase tickets to play the game only at a finite, and possibly quite modest, entry fee.
- Why? Because investors do not assign the same value per dollar to all payoffs. This gives rise to the idea of diminishing marginal utility.

Figure 6A.1 Diminishing marginal utility



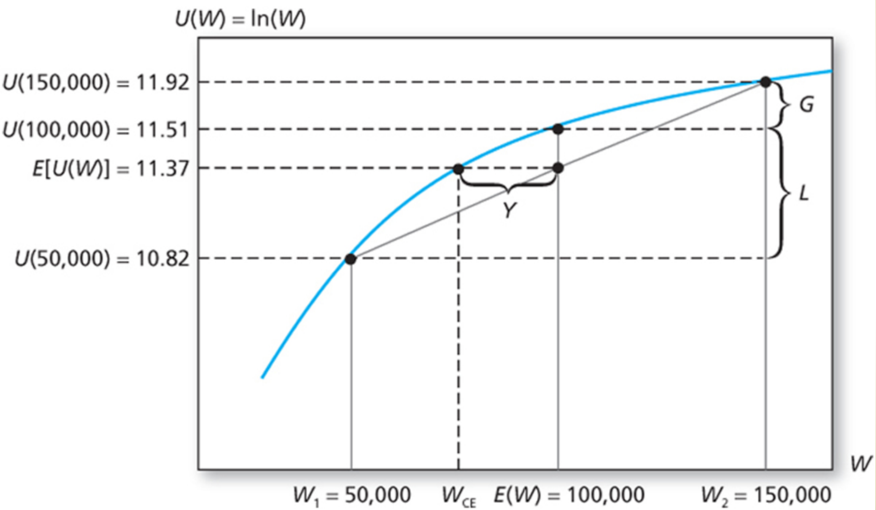
Certainty Equivalent

- Consider the following simple prospect:



- Suppose, that the curve in Figure 6A.1 represents the investor's utility value of wealth. Such a curve can be represented with a log function.

Figure 6A.2 Fair games and expected utility



Fair games and expected utility

- Figure 6A.2 shows that the loss in utility from losing \$50,000 exceeds the gain from winning \$50,000.
  - With the increase in wealth utility increases from $ln(100,000) = 11.51$ to $ln(150,000) = 11.92$ (or by 0.41)
  - With the decrease in wealth utility decreases from $ln(100,000) = 11.51$ to $ln(50,000) = 10.82$ (or by 0.69)
- the expected utility is:

$$E[U(W)] = pU(W_1) + (1-p)U(W_2) = \frac{1}{2}ln(50,000) + \frac{1}{2}ln(150,000) = 11.37$$

Certainty equivalent

- If the prospect is rejected, the utility value of the (sure) $100,000 is $ln(100,000) = 11.51$, greater than that of the fair game (11.37). Hence the risk-averse investor will reject the fair game.

- Certainty equivalent value — the amount that, if received with certainty, the investor would consider equally attractive as the risky prospect.

$$W_{CE} = ?$$

$$\begin{aligned} ln(W_{CE}) &= 11.37 \\ W_{CE} &= exp(ln(W_{CE})) \\ &= exp(11.37) \\ &= \$86,681.87 \end{aligned}$$

Concept Check

CONCEPT CHECK 6A.1

Suppose the utility function is $U(W) = \sqrt{W}$

a. What is the utility level at wealth levels $50,000 and $150,000?

b. What is expected utility if *p* still equals .5?

c. What is the certainty equivalent of the risky prospect?

d. Does this utility function also display risk aversion?

e. Does this utility function display more or less risk aversion than the log utility function?