Chapter 7, Optimal Risky Portfolios

Alexander Philipov

George Mason University | School of Business

Chapter 7 Overview

The Investment Decision

Diversification

Two Risky Assets

Stocks, Bonds, and Bills

Markowitz Optimization

Risk Pooling, Risk Sharing, and the Long Run

The Investment Decision

- Top-down process with 3 steps:
  1. Capital allocation between the risky portfolio and risk-free asset
  2. Asset allocation across broad asset classes
  3. Security selection of individual assets within each asset class

Diversification and Porftolio Risk

- Market risk
  - Risk attributable to marketwide risk sources and remains even after extensive diversification
  - Also call systematic or nondiversifiable
- Firm-specific risk
  - Risk that can be eliminated by diversification
  - Also called diversifiable or nonsystematic

Figure 7.1 Porfolio Risk and the Number of Stocks

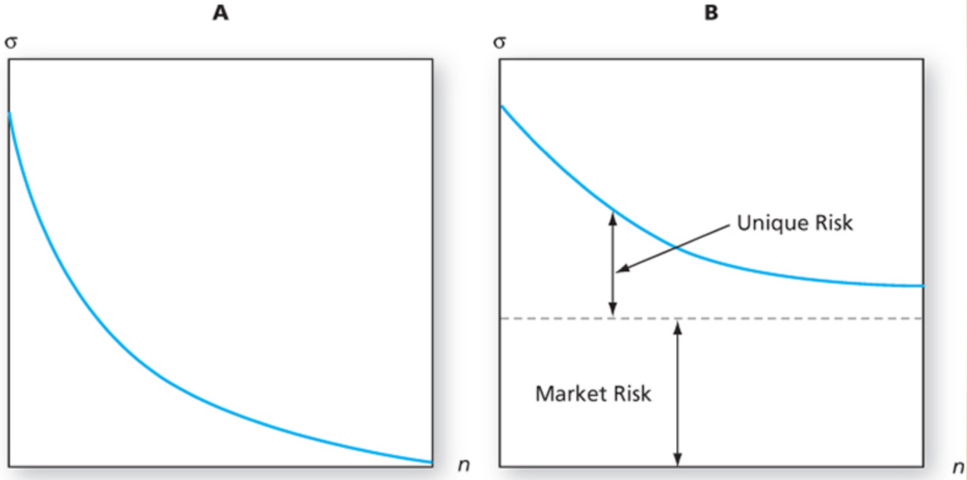
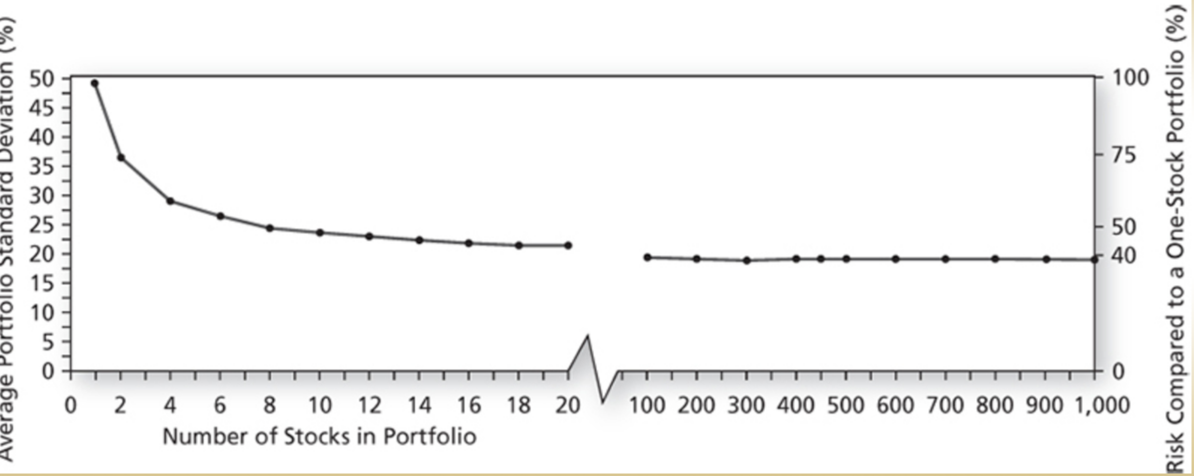


Figure 7.2 Portfolio Diversification



Portfolio of Two Risky Assets

- Portfolio risk (variance) depends on the correlation between the returns of the assets in the portfolio
- Covariance and the correlation coefficient provide a measure of the way returns of two assets move together (covary)

Table 7.1 Descriptive Statistics

Table 7.1

Descriptive statistics for two mutual funds

	Debt		Equity
Expected return, $E(r)$	8%		13%
Standard deviation, σ	12%		20%
Covariance, $\text{Cov}(r_D, r_E)$		72	
Correlation coefficient, ρ_{DE}		.30	

Spreadsheet available:

`http://somfin.gmu.edu/courses/mba706/excel/BKM_10e_Ch07_Two_Security_Model.xls`

Portfolio of Two Risky Assets: Return

- Portfolio return:

$$r_P = w_D r_D + w_E r_E \tag{7.1}$$

where

$$w_D = \text{Bond weight}$$
$$r_D = \text{Bond return}$$
$$w_E = \text{Equity weight}$$
$$r_E = \text{Equity return}$$

- Portfolio expected return:

$$E(r_P) = w_D E(r_D) + w_E E(r_E) \tag{7.2}$$

Portfolio of Two Risky Assets: Risk

- Portfolio variance:

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E) \quad (7.3)$$

where

$$\sigma_D^2 = \text{Bond variance}$$
$$\sigma_E^2 = \text{Equity variance}$$
$$Cov(r_D, r_E) = \text{Covariance of returns for bond and equity}$$

Portfolio of Two Risky Assets: Covariance

- Covariance of returns on bond and equity:

$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E \qquad (7.6)$$

where

$$\rho_{DE} = \text{Correlation coefficient of returns}$$
$$\sigma_D = \text{Standard deviation of bond returns}$$
$$\sigma_E = \text{Standard deviation of equity returns}$$

Table 7.2 Calculating Portfolio Variance

Table 7.2

Computation of portfolio variance from the covariance matrix

A. Bordered Covariance Matrix

Portfolio Weights	w_D	w_E
w_D	$\text{Cov}(r_D, r_D)$	$\text{Cov}(r_D, r_E)$
w_E	$\text{Cov}(r_E, r_D)$	$\text{Cov}(r_E, r_E)$

B. Border-Multiplied Covariance Matrix

Portfolio Weights	w_D	w_E
w_D	$w_D w_D \text{Cov}(r_D, r_D)$	$w_D w_E \text{Cov}(r_D, r_E)$
w_E	$w_E w_D \text{Cov}(r_E, r_D)$	$w_E w_E \text{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D)$	$w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$
Portfolio variance	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D) + w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$	

Concept check

CONCEPT CHECK 7.1

a. First confirm for yourself that our simple rule for computing the variance of a two-asset portfolio from the bordered covariance matrix is consistent with Equation 7.3.

b. Now consider a portfolio of three funds, X, Y, Z, with weights w_X, w_Y, and w_Z. Show that the portfolio variance is

$$w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + w_Z^2\sigma_Z^2 + 2w_Xw_Y \text{Cov}(r_X, r_Y) + 2w_Xw_Z \text{Cov}(r_X, r_Z) + 2w_Yw_Z \text{Cov}(r_Y, r_Z)$$

Portfolio of Two Risky Assets: Correlation

- Range of values for ρ_{DE}:

$$-1.0 \leq \rho_{DE} \leq +1.0$$

- If $\rho = 1.0$, the securities are pefectly positively correlated
- If $\rho = -1.0$, the securities are pefectly negatively correlated

Portfolio of Two Risky Assets: Correlation

- When $\rho_{DE} = 1$, there is no diversification

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \sigma_D \sigma_E) \quad (7.7)$$

$$= (w_D\sigma_D + w_E\sigma_E)^2 \quad (7.8)$$

$$\sigma_P = w_D\sigma_D + w_E\sigma_E \quad (7.9)$$

  - σ_P is a weighted average of component standard deviations

- When $\rho_{DE} = -1$, there is potential for perfect hedge

$$\sigma_P^2 = (w_D\sigma_D - w_E\sigma_E)^2 \quad (7.10)$$

solve for $(w_D\sigma_D - w_E\sigma_E) = 0$:

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D \quad (7.12)$$

Example 7.1

- Using data from Table 7.1:

$$
\begin{aligned}
E(r_P) &= 8w_D + 13w_D \\
\sigma_P^2 &= 12^2 w_D^2 + 20^2 w_E^2 + 2 \times 12 \times 20 \times w_D w_E \\
&= 144 w_D^2 + 400 w_E^2 + 144 w_D w_E \\
\sigma_P &= \sqrt{\sigma_P^2}
\end{aligned}
$$

- Experiment with different proportions of w_D and w_E. See Table 7.3 on next slide.

Table 7.3

			Portfolio Standard Deviation for Given Correlation			
w_D	w_E	$E(r_P)$	$\rho = -1$	$\rho = 0$	$\rho = .30$	$\rho = 1$
0.00	1.00	13.00	20.00	20.00	20.00	20.00
0.10	0.90	12.50	16.80	18.04	18.40	19.20
0.20	0.80	12.00	13.60	16.18	16.88	18.40
0.30	0.70	11.50	10.40	14.46	15.47	17.60
0.40	0.60	11.00	7.20	12.92	14.20	16.80
0.50	0.50	10.50	4.00	11.66	13.11	16.00
0.60	0.40	10.00	0.80	10.76	12.26	15.20
0.70	0.30	9.50	2.40	10.32	11.70	14.40
0.80	0.20	9.00	5.60	10.40	11.45	13.60
0.90	0.10	8.50	8.80	10.98	11.56	12.80
1.00	0.00	8.00	12.00	12.00	12.00	12.00
			Minimum Variance Portfolio			
		w_D	0.6250	0.7353	0.8200	—
		w_E	0.3750	0.2647	0.1800	—
		$E(r_P)$	9.8750	9.3235	8.9000	—
		σ_P	0.0000	10.2899	11.4473	—

Table 7.3

Expected return and standard deviation with various correlation coefficients

Figure 7.3 Portfolio Expected Return

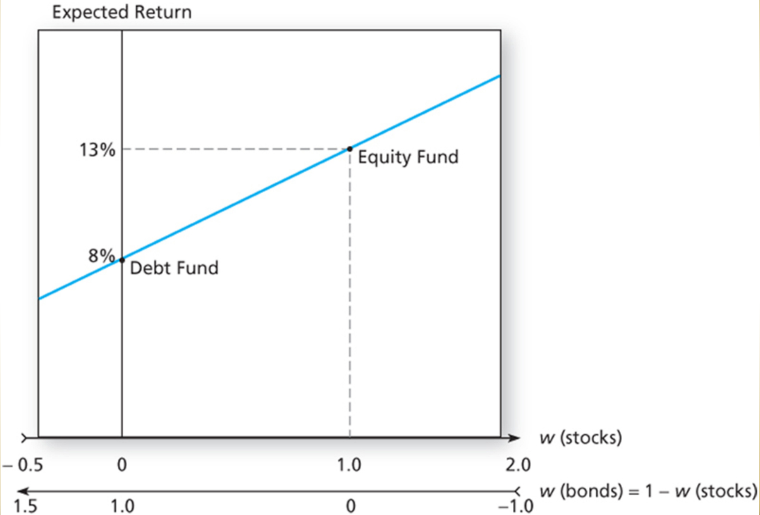


Figure 7.4 Portfolio Standard Deviation

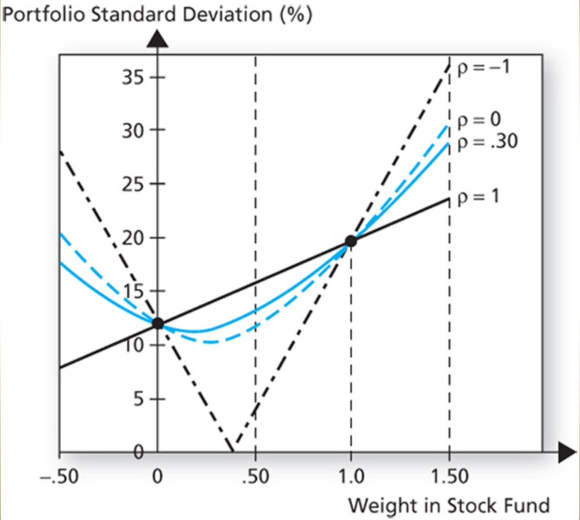
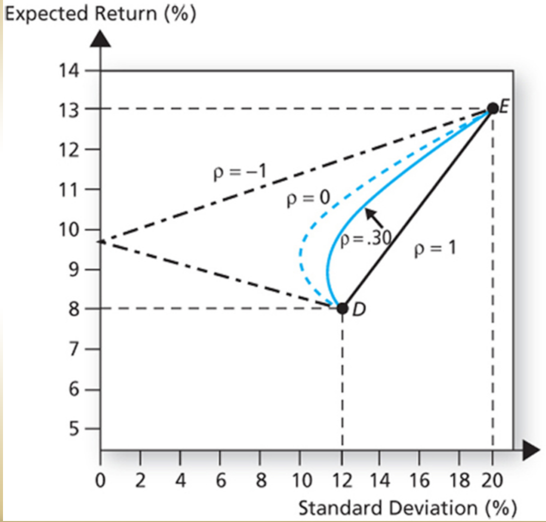


Figure 7.5 Expected Return As a Function of SD



The Minimum Variance Portfolio

- The minimum variance portfolio is the portfolio composed of the risky assets that has the smallest standard deviation; the portfolio with least risk
- For the two-asset case the weights of the min-var portfolio are:

$$w_D^{min} = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)} \qquad w_E^{min} = 1 - w_D^{min}$$

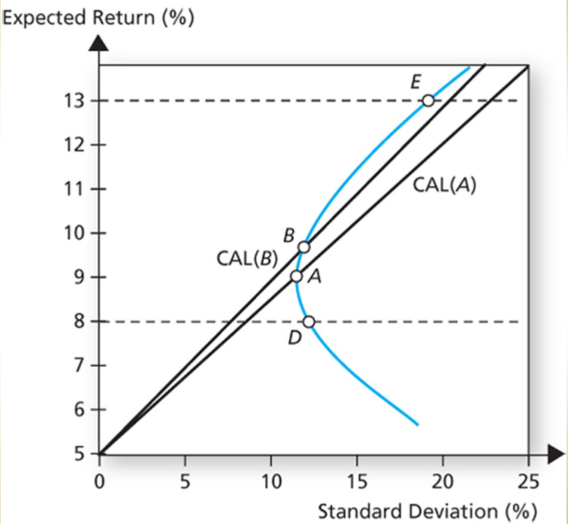
- The amount of possible risk reduction through diversification depends on the correlation:
  - If $\rho = +1.0$, no risk reduction is possible
  - If $\rho = 0$, σ_P may be less than the standard deviation of either component asset
  - If $\rho = -1.0$, a riskless hedge is possible

Concept check

CONCEPT CHECK 7.2

Compute and draw the portfolio opportunity set for the debt and equity funds when the correlation coefficient between them is $\rho = .25$.

Figure 7.6 The Opportunity Set with Stocks and Bonds



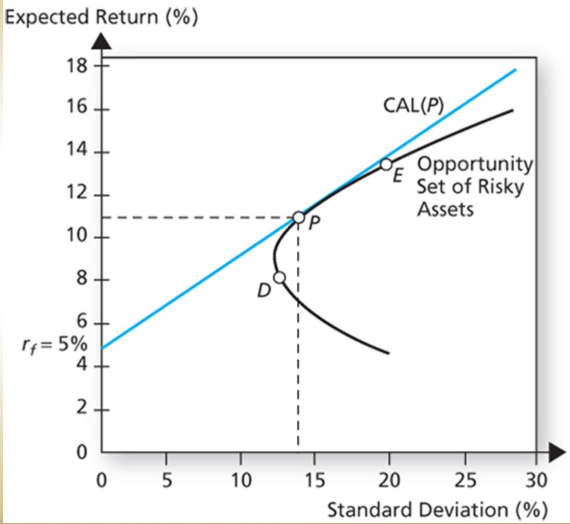
The Sharpe Ratio

- Maximize the slope of the CAL for any possible portfolio, *P*
- The objective function is the slope:

$$S = \frac{E(r_P) - r_f}{\sigma_P}$$

- The slope is also the Sharpe ratio

Figure 7.7 Optimal Risky Portfolio



Optimal Risky Portfolio Problem

$$\max_{w_i} S_P = \frac{E(r_P) - r_f}{\sigma_P}$$

$$w_D = \frac{E(R_D)\sigma_E^2 - E(R_E)Cov(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]Cov(R_D, R_E)} \tag{7.13}$$

$$w_E = 1 - w_D$$

Example 7.2 Optimal Risky Portfolio

Using our data, the optimal risky portfolio is:

$$w_D = \frac{(8-5)400 - (13-5)72}{(8-5)400 + (13-5)144 - (8-5+13-5)72} = 0.40$$

$$w_E = 1 - 0.40 = 0.60$$

giving an expected return and standard deviation:

$$E(r_P) = (0.4 \times 8) + (0.6 \times 13) = 11\%$$

$$\sigma_P = \sqrt{(0.4^2 \times 144) + (0.6^2 \times 400) + (2 \times 0.4 \times 0.6 \times 72)} = 14.2\%$$

and a slope of the CAL(P):

$$S_P = \frac{11-5}{14.2} = 0.42$$

Example 7.3 Optimal Complete Portfolio

Now that asset allocation is decided, we can find each investor's optimal capital allocation. An investor with a coefficient of risk aversion A = 4 would take a position in portfolio P (note the use of decimals):

$$y = \frac{E(r_P) - r_f}{A\sigma_P^2} = \frac{0.11 - 0.05}{4 \times 0.142^2} = 0.7439 \tag{7.14}$$

Thus the investor will invest 74.39% of his or her wealth in portfolio P and 25.61% in T-bills. Portfolio P consists of 40% in bonds, so the fraction of wealth in bonds will be $y_{w_D} = 0.4 \times 0.7439 = 0.2976$, or 29.76%. Similarly, the investment in stocks will be $y_{w_E} = 0.6 \times 0.7439 = 0.4463$, or 44.63%. The graphical solution of this asset allocation problem is in Figs 7.8 and 7.9.

Figure 7.8 Determining Overall Optimal Portfolio

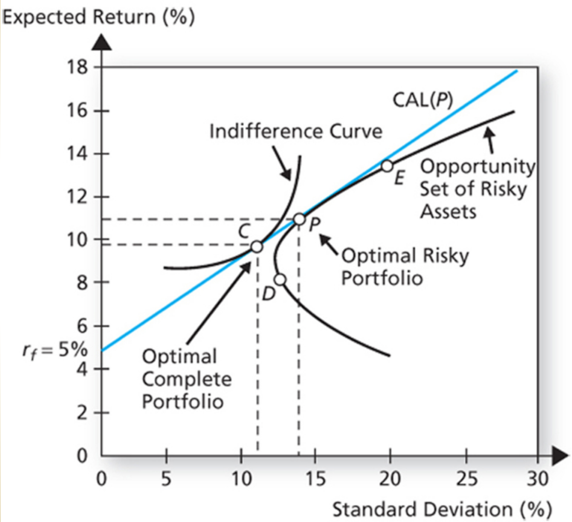
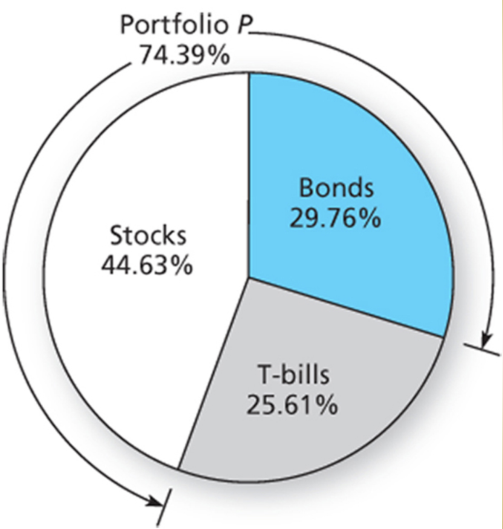


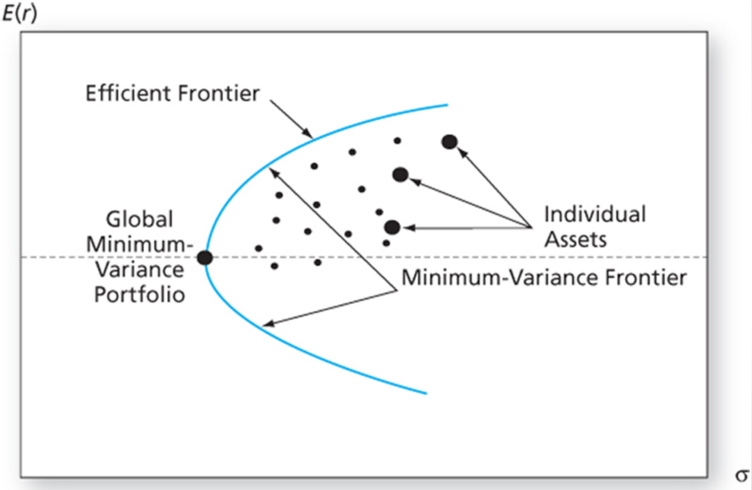
Figure 7.9 The Proportions of the Optimal Complete Portfolio



Markowitz Portfolio Optimization Model

- Security selection
  - The first step is to determine the risk-return opportunities available
  - All portfolios that lie on the minimum-variance frontier from the global minimum-variance portfolio and upward provide the best risk-return combinations

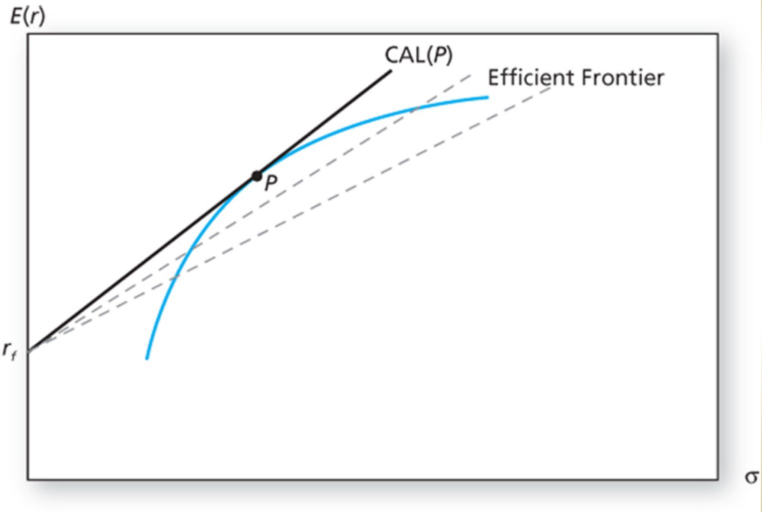
Figure 7.10 The Minimum-Variance Frontier of Risky Assets



Markowitz Portfolio Optimization Model

- Search for the CAL with the highest reward-to-variability ratio
- Everyone invests in *P*, regardless of their degree of risk aversion
    - More risk averse investors put more in the risk-free asset
    - Less risk averse investors put more in *P*

Figure 7.11 Efficient Frontier of Risky Assets with the Optimal CAL



Markowitz Portfolio Optimization Model

- Capital Allocation and the Separation Property
  - Portfolio choice problem may be separated into two independent tasks
    - Determination of the optimal risky portfolio is purely technical
    - Allocation of the complete portfolio to risk-free versus the risky portfolio depends on personal preference

Figure 7.12 The Efficient Portfolio Set

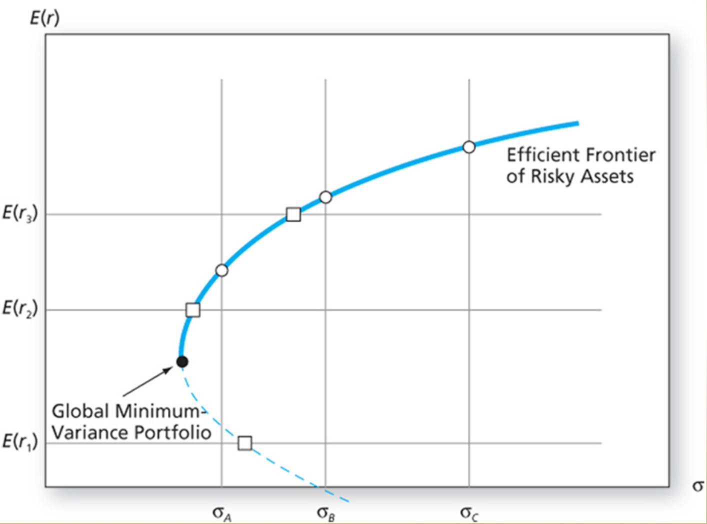
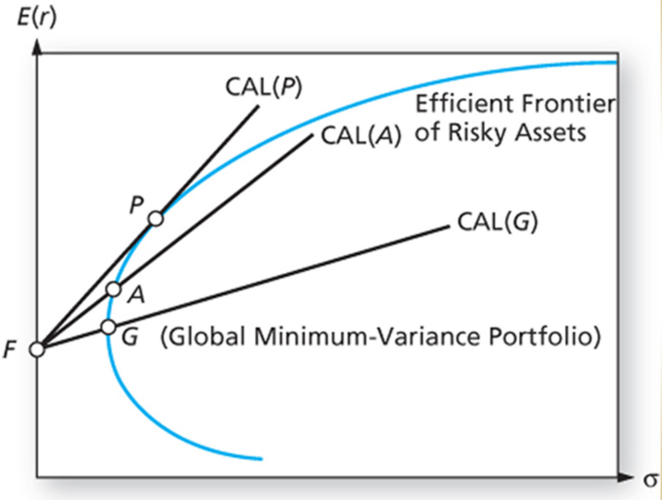


Figure 7.13 CAL with Various Portfolios from the Efficient Set



The Power of Diversification

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j Cov(r_i, r_j) \tag{7.16}$$

We can separate into average variance and average covariance. To simplify, assume equal weights:

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{1}{n} \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{n^2} Cov(r_i, r_j) \tag{7.17}$$

and we can express portfolio variance as:

$$\sigma_P^2 = \frac{1}{n} \bar{\sigma}^2 + \frac{n-1}{n} \overline{Cov} \tag{7.20}$$

Table 7.4 Risk Reduction of Equally Weighted Portfolios

Table 7.4

Risk reduction of equally weighted portfolios in correlated and uncorrelated universes

		$\rho = 0$		$\rho = .40$	
Universe Size n	Portfolio Weights $w = 1/n$ (%)	Standard Deviation (%)	Reduction in σ	Standard Deviation (%)	Reduction in σ
1	100	50.00	14.64	50.00	8.17
2	50	35.36		41.83	
5	20	22.36	1.95	36.06	0.70
6	16.67	20.41		35.36	
10	10	15.81	0.73	33.91	0.20
11	9.09	15.08		33.71	
20	5	11.18	0.27	32.79	0.06
21	4.76	10.91		32.73	
100	1	5.00	0.02	31.86	0.00
101	0.99	4.98		31.86	

The Power of Diversification

- Further simplify, same standard deviation across all stocks, σ:

$$\sigma_P^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2 \tag{7.21}$$

- Portfolio variance can be driven to zero if the average covariance is zero (only firm specific risk)
- The irreducible risk of a diversified portfolio depends on the covariance of the returns, which is a function of the systematic factors in the economy

Markowitz Portfolio Optimization Model

- Optimal Portfolios and Nonnormal Returns
  - Fat-tailed distributions can result in extreme values of VaR and ES and encourage smaller allocations to the risky portfolio
  - If other portfolios provide sufficiently better VaR and ES values than the mean-variance efficient portfolio, we may prefer these when faced with fat-tailed distributions

Risk Pooling and the Insurance Principle

- Risk pooling
  - Merging uncorrelated, risky projects as a means to reduce risk
  - It increases the scale of the risky investment by adding additional uncorrelated assets
- The insurance principle
  - Risk increases less than proportionally to the number of policies when the policies are uncorrelated
  - Sharpe ratio increases

Risk pooling portfolio, R_Z

$$R_Z = yR + yR + (1-2y) = 2yR \qquad \text{(double } R_P\text{)}$$

$$\sigma_Z^2 = y^2\sigma^2 + y^2\sigma^2 + 0 = 2y^2\sigma^2 \qquad \text{(double the variance of } P\text{)}$$

$$\sigma_Z = \sqrt{\sigma_Z^2} = y\sigma\sqrt{2} \qquad (\sqrt{2} = 1.41 \times \text{ s.d. of } P)$$

$$S_Z = R_Z/\sigma_Z = 2yR/(y\sigma\sqrt{2}) = \sqrt{2}R/\sigma \qquad (\sqrt{2} = 1.41 \times \text{ Sharpe ratio of } P)$$

Risk Sharing

- As risky assets are added to the portfolio, a portion of the pool is sold to maintain a risky portfolio of fixed size
- Risk sharing combined with risk pooling is the key to the insurance industry
- True diversification means spreading a portfolio of fixed size across many assets, not merely adding more risky bets to an ever-growing risky portfolio

Risk sharing portfolio, R_V

$$R_Z = yR + yR + (1 - 2y) = 2yR$$
$$\sigma_Z^2 = y^2\sigma^2 + y^2\sigma^2 + 0 = 2y^2\sigma^2$$
$$\sigma_Z = \sqrt{\sigma_Z^2} = y\sigma\sqrt{2}$$
$$S_Z = R_Z/\sigma_Z = 2yR/(y\sigma\sqrt{2}) = \sqrt{2}R/\sigma$$

$$R_V = 2(y/2)R = yR$$
$$\sigma_V^2 = 2(y/2)^2\sigma^2 = y^2\sigma^2/2$$
$$\sigma_V = \sqrt{\sigma_V^2} = y\sigma/\sqrt{2}$$
$$S_V = R_V/\sigma_V = \sqrt{2}R/\sigma$$

Investment for the Long Run

- Long-Term Strategy
  - Invest in the risky portfolio for 2 years
    - Long-term strategy is riskier
    - Risk can be reduced by selling some of the risky assets in year 2
    - "Time diversification" is not true diversification
- Short-Term Strategy
  - Invest in the risky portfolio for 1 year and in the risk-free asset for the second year

Spread risky budget over two periods

1. Invest the whole budget at risk for one period, and then withdraw the entire proceeds, placing them in a risk-free asset in the other period. Because you are invested in the risky asset for only 1 year, the risk premium over the whole investment period is R, the 2-year SD is σ, and the 2-year Sharpe ratio is $S = R/\sigma$.

Spread risky budget over two periods

2. Invest the whole budget in the risky asset for both periods. The 2-year risk premium is $2R$ (assuming continuously compounded rates), the 2-year variance is $2\sigma^2$, the 2-year SD is $\sigma\sqrt{2}$, and the 2-year Sharpe ratio is $SR = R\sqrt{2}/\sigma$. This is analogous to risk pooling, taking two "bets" on the risky portfolio instead of one (as in Strategy 1).

Spread risky budget over two periods

3. Invest half the investment budget in the risky position in each of two periods, placing the remainder of funds in the risk-free asset. The 2-year risk premium is R, the 2-year variance is $2(1/2\sigma)^2 = \sigma^2/2$, the SD is $\sigma/\sqrt{2}$ and the Sharpe ratio is $SR = R\sqrt{2}/\sigma$. This is analogous to risk sharing, taking a fractional position in each year's investment return.

Spreadsheet 7.B1

	A	B	C	D	E	F	G
1							
2			Scenario rates of return				
3	Scenario	Probability	$r_D(i)$	$r_D(i) + 0.03$	$0.4*r_D(i)$		
4	1	0.14	–0.10	–0.07	–0.040		
5	2	0.36	0.00	0.03	0.000		
6	3	0.30	0.10	0.13	0.040		
7	4	0.20	0.32	0.35	0.128		
8		Mean	0.080	0.110	0.032		
9		Cell C8	=SUMPRODUCT($B$4:$B$7,C4:C7)				
10							
11							
12							

Spreadsheet 7.B2

	H	I	J	K	L
1					
2			Scenario rates of return		Portfolio return
3	Scenario	Probability	$r_D(i)$	$r_E(i)$	$0.4*r_D(i)+0.6*r_E(i)$
4	1	0.14	−0.10	−0.35	−0.2500
5	2	0.36	0.00	0.20	0.1200
6	3	0.30	0.10	0.45	0.3100
7	4	0.20	0.32	−0.19	0.0140
8		Mean	0.08	0.12	0.1040
9		Cell L4	=0.4*J4+0.6*K4		
10		Cell L8	=SUMPRODUCT($I$4:$I$7,L4:L7)		
11					
12					

Spreadsheet 7.B3

	A	B	C	D	E	F	G
13							
14			Scenario rates of return				
15	Scenario	Probability	$r_D(i)$	$r_D(i) + 0.03$	$0.4 * r_D(i)$		
16	1	0.14	–0.10	–0.07	–0.040		
17	2	0.36	0.00	0.03	0.000		
18	3	0.30	0.10	0.13	0.040		
19	4	0.20	0.32	0.35	0.128		
20		Mean	0.0800	0.1100	0.0240		
21		Variance	0.0185	0.0185	0.0034		
22		SD	0.1359	0.1359	0.0584		
23	Cell C21	=SUMPRODUCT($B$16:$B$19,C16:C19,C16:C19)–C20^2					
24	Cell C22	=C21^0.5					

Spreadsheet 7.B4

	A	B	C	D	E	F	G	H
1		Rates of Return			Deviation from Mean			Product of
2	Probability	Bonds	Stocks		Bonds	Stocks		Deviations
3	0.25	–2	30		–8	20		–160
4	0.50	6	10		0	0		0
5	0.25	14	–10		8	–20		–160
6	Mean:	6	10		0	0		–80

Spreadsheet 7.B5

	H	I	J	K	L	M
13						
14			Scenario rates of return			
15	Scenario	Probability	$r_D(i)$	$r_E(i)$		
16	1	0.14	–0.10	–0.35		
17	2	0.36	0.00	0.20		
18	3	0.30	0.10	0.45		
19	4	0.20	0.32	–0.19		
20		Mean	0.08	0.12		
21		SD	0.1359	0.2918		
22		Covariance	–0.0034			
23		Correlation	–0.0847			
24	Cell J22	=SUMPRODUCT(I16:I19,J16:J19,K16:K19)–J20*K20				
25	Cell J23	=J22/(J21*K21)				

Spreadsheet 7.B6

	A	B	C	D	E	F	G
25							
26							
27							
28			Scenario rates of return		Portfolio return		
29	Scenario	Probability	$r_D(i)$	$r_E(i)$	$0.4*r_D(i)+0.6r_E(i)$		
30	1	0.14	−0.10	−0.35	−0.25		
31	2	0.36	0.00	0.20	0.12		
32	3	0.30	0.10	0.45	0.31		
33	4	0.20	0.32	−0.19	0.014		
34		Mean	0.08	0.12	0.1040		
35		SD	0.1359	0.2918	0.1788		
36		Covariance	−0.0034		SD: 0.1788		
37		Correlation	−0.0847				
38	Cell E35 =SUMPRODUCT(B30:B33,E30:E33,E30:E33)−E34^2)^0.5						
39	Cell E36 =(0.4*C35)^2+(0.6*D35)^2+2*0.4*0.6*C36)^0.5						

Figure 7.1U

	A	B	C	D	E	F
1	**Asset Allocation Analysis: Risk and Return**					
2		**Expected**	**Standard**	**Correlation**		
3		**Return**	**Deviation**	**Coefficient**	**Covariance**	
4	Security 1	0.08	0.12	0.3	0.0072	
5	Security 2	0.13	0.2			
6	T-Bill	0.05	0			
7						
8	**Weight**	**Weight**		**Expected**	**Standard**	**Reward to**
9	**Security 1**	**Security 2**		**Return**	**Deviation**	**Volatility**
10	1	0		0.08000	0.12000	0.25000
11	0.9	0.1		0.08500	0.11559	0.30281
12	0.8	0.2		0.09000	0.11454	0.34922
13	0.7	0.3		0.09500	0.11696	0.38474
14	0.6	0.4		0.10000	0.12264	0.40771

Figure 7.2U

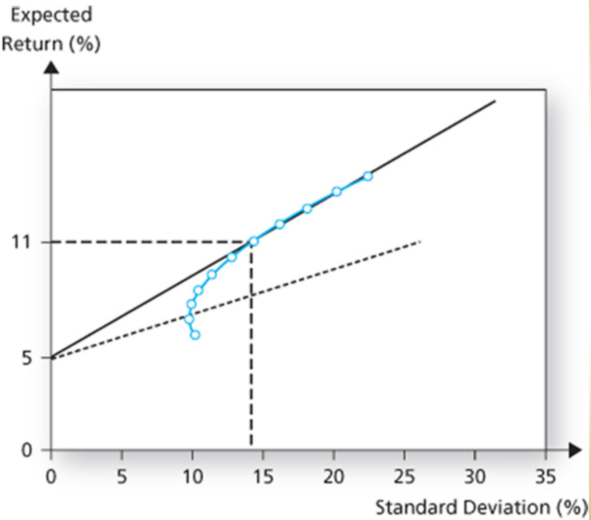


Table 7.3U

	A	B	C	D	E	F
1	Efficient Frontier for World Equity Benchmark Securities (WEBS)					
2						
3		Mean	Standard			
4	WEBS	Return	Deviation	Country		
5	EWD	15.5393	26.4868	Sweden		
6	EWH	6.3852	41.1475	Hong Kong		
7	EWI	26.5999	26.0514	Italy		
8	EWJ	1.4133	26.0709	Japan		
9	EWL	18.0745	21.6916	Switzerland		
10	EWP	18.6347	25.0779	Spain		
11	EWW	16.2243	38.7686	Mexico		
12	S&P 500	17.2306	17.1944			